DOCUMENT RESUME

ED 173 149 SE 028 244

AUTHOR Allen, Frank B.; And Others
TITLE Mathematics for High School, Intermediate Mathematics (Part 3). Commentary for Teachers. Revised Edition.
INSTITUTION Stanford Univ., Calif. School Mathematics Study Group.
SPONS AGENCY National Science Foundation, Washington, D.C.
PUB DATE 60
NOTE 158p.; For related documents, see SE 028 243 and ED 135 628; Contains occasional light and broken type

EDRS PRICE MF01/PC07 Plus Postage.
DESCRIPTORS *Algebra; Curriculum; *Curriculum Guides; *Instruction; Mathematics Education; *Number Concepts; Probability; Secondary Education; *Secondary School Mathematics
IDENTIFIERS *School Mathematics Study Group; *Vectors (Mathematics)

ABSTRACT

This is part three of a three-part manual for teachers using SMSG high school text materials. Each chapter contains a commentary on the text, answers to the exercises, and a set of illustrative test questions. Chapter topics include: (1) the system of vectors; (2) polar form of complex numbers; (3) sequences and series; (4) permutations; (5) combinations, and the binomial theorem; and (6) algebraic structures. (MP)

**********
* Reproductions supplied by EDRS are the best that can be made *
* from the original document. *
**********

# MATHEMATICS FOR HIGH SCHOOL

## INTERMEDIATE MATHEMATICS (Part 3)

## *Commentary for Teachers*

(revised edition)

Prepared under the supervision of the Panel on Sample Textbooks of the School Mathematics Study Group:

Frank B. Allen, Lyons Township High School

Edwin C. Douglas, Taft School

Donald E. Richmond, Williams College

Charles E. Rickart, Yale University

Henry Swain, New Trier Township High School

Robert J. Walker, Cornell University

*Financial support for the School Mathematics Study Group has been provided by the National Science Foundation.*

*Copyright 1960 by Yale University.*
PRINTED IN THE U.S.A.

CONTENTS


| Chapter | | | Page |
|---|---|---|---|
| 11. | THE SYSTEM OF VECTORS | | 705 |
| | 11- 0. | Introduction | 705 |
| | 11- 1. | Directed Line Segments | 705 |
| | 11- 2. | Applications to Geometry | 707 |
| | 11- 3. | Vectors and Scalars; Components | 710 |
| | 11- 4. | Inner Product | 714 |
| | 11- 5. | Applications of Vectors in Physics | 717 |
| | 11- 6. | Vectors as a Formal Mathematical System | 737 |
| | 11- 7. | Illustrative Test Questions | 739 |
| 12. | POLAR FORM OF COMPLEX NUMBERS | | 747 |
| | 12- 1. | Introduction | 747 |
| | 12- 2. | Products and Polar Form | 750 |
| | 12- 3. | Integral Powers; Theorem of deMoivre | 752 |
| | 12- 4. | Square Roots | 754 |
| | 12- 5. | Quadratic Equations with Complex Coefficients | 754 |
| | 12- 6. | Roots of Order n | 755 |
| | 12- 7. | Miscellaneous Exercises - Answers | 757 |
| | 12- 8. | Suggested Test Items | 759 |
| 13. | SEQUENCES AND SERIES | | 765 |
| | 13- 1. | Introduction | 765 |
| | 13- 2. | Arithmetic Sequences and Series | 767 |
| | 13- 3. | Geometric Sequences and Series | 771 |
| | 13- 4. | Limit of a Sequence | 774 |
| | 13- 5. | Sum of an Infinite Series | 778 |
| | 13- 6. | The Infinite Geometric Series | 787 |
| | | Miscellaneous Exercises - Answers | 791 |
| | | Illustrative Test Questions | 797 |
| 14. | PERMUTATIONS, COMBINATIONS, AND THE BINOMIAL THEOREM | | 809 |
| | 14- 1. | Introduction, Counting Problems | 809 |
| | 14- 2. | Ordered Multiples | 813 |
| | 14- 3. | Permutations | 816 |
| | 14- 4. | Combinations | 820 |
| | 14- 5. | The Binomial Theorem | 823 |
| | 14- 6. | Arrangements | 827 |
| | 14- 7. | Selections with Repetitions | 828 |
| | 14- 8. | Miscellaneous Exercises - Answers | 830 |
| | 14- 9. | Illustrative Test Questions | 833 |

Chapter Page

15. ALGEBRAIC STRUCTURES . . . . . 841
15- 2. Internal Operation . . . . . 841
15- 3. Group . . . . . 841
15- 4. Some General Properties of Groups . . . . 842
15- 5. An Example of a Non-Abelian Group . . . . 848
15- 6. Field . . . . . 850
15- 7. Subfield . . . . . 854

Commentary for Teachers

Chapter 11

# THE SYSTEM OF VECTORS

## 11-0. Introduction.

Vectors have both a geometric and algebraic aspect. The first part of the text, 11-1 and 11-2, is primarily geometric. The algebra of directed line segments is considered to be a pleasant device for solving geometric problems. In Sections 11-3 and 11-4 the algebra of vectors is worked out more carefully. Section 11-5 is about applications of vectors to physics. While this kind of discussion helped form the whole subject originally, it no longer is the central topic in vector studies. Section 11-6 is concerned with the system of vectors as a whole. Instead of examining individual vectors the student is exposed here to statements about all vectors.

---

## 11-1. Directed Line Segments.

The main ideas of this section are equivalence of directed line segments, addition of directed line segments, and multiplication of directed line segments by real numbers. The student is required to translate statements of geometric relation into algebraic language.

### Exercises 11-1. Answers.

1. $\overrightarrow{AA}$ , $\overrightarrow{AB}$ , $\overrightarrow{BA}$ , $\overrightarrow{BB}$ .

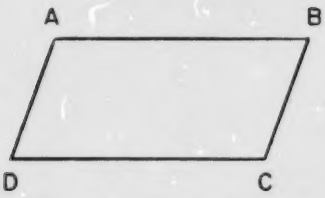
2. $\overrightarrow{AA}$ , $\overrightarrow{AB}$ , $\overrightarrow{AC}$ , $\overrightarrow{BB}$ , $\overrightarrow{BA}$ , $\overrightarrow{BC}$ , $\overrightarrow{CC}$ , $\overrightarrow{CB}$ , $\overrightarrow{CA}$ .

This is true whether the points are collinear or not.

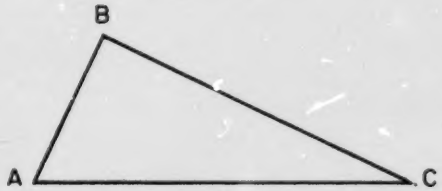
3. $\overrightarrow{AA} \doteq \overrightarrow{BB} \doteq \overrightarrow{CC} \doteq \overrightarrow{DD}$

$\overrightarrow{AB} \doteq \overrightarrow{DC}$ , $\overrightarrow{BA} \doteq \overrightarrow{CD}$

$\overrightarrow{AD} \doteq \overrightarrow{BC}$ , $\overrightarrow{DA} \doteq \overrightarrow{CB}$




$\overrightarrow{AC}$ , $\overrightarrow{CA}$ , $\overrightarrow{BD}$ , $\overrightarrow{DB}$ are also included in the list of directed line segments. From plane geometry the diagonals of a parallelogram have equal measure. This might lead one to say $\overrightarrow{AC}$ and $\overrightarrow{BD}$ are equivalent. One needs to turn again to the Definition 11-1a for equivalent directed line segments. The same consideration can be invoked to convince one that $\overrightarrow{AC}$ and $\overrightarrow{CA}$ are not equivalent.




4. (a) $\overrightarrow{AC}$

(b) $\overrightarrow{AC}$

(c) $\overrightarrow{AC}$

(d) $\overrightarrow{BA}$

(e) $\overrightarrow{AA}$ , for $(\overrightarrow{AB} + \overrightarrow{BC}) + \overrightarrow{CA} = \overrightarrow{AC} + \overrightarrow{CA} = \overrightarrow{AA}$ .

(f) $\overrightarrow{BB}$ , for $\overrightarrow{BA} + (\overrightarrow{AC} + \overrightarrow{CB}) = \overrightarrow{BA} + \overrightarrow{BA} = \overrightarrow{BB}$ .

(g) $\overrightarrow{CB} + \overrightarrow{CA}$ . Consider what must be added to $\overrightarrow{AC}$ to give $\overrightarrow{CB}$.

$_____ + \overrightarrow{AC} = \overrightarrow{CB}$

$\overrightarrow{CB} + \overrightarrow{CA} + \overrightarrow{AC} = \overrightarrow{CB}$ .

5. (a) A ——— X ——— B $\overrightarrow{AX} = \frac{1}{2}\overrightarrow{AB}$ , $\overrightarrow{BX} = \frac{1}{2}\overrightarrow{BA}$ , $r = \frac{1}{2}$ , $s = \frac{1}{2}$ .

(b) A ——— B ——— X $\overrightarrow{AX} = 2\overrightarrow{AB}$ , $\overrightarrow{BX} = -1\overrightarrow{BA}$ , $r = 2$ , $s = -1$ .

(c) B ——— A ——— X $\overrightarrow{AX} = -1\overrightarrow{AB}$ , $\overrightarrow{BX} = 2\overrightarrow{BA}$ , $r = -1$, $s = 2$ .

(d) A ——— X ——— B $\overrightarrow{AX} = \frac{2}{3}\overrightarrow{AB}$ , $\overrightarrow{BX} = \frac{1}{3}\overrightarrow{BA}$ , $r = \frac{2}{3}$ , $s = \frac{1}{3}$ .

(e) A ——— B ——— X $\overrightarrow{AX} = \frac{3}{2}\overrightarrow{AB}$ , $\overrightarrow{BX} = -\frac{1}{2}\overrightarrow{BA}$ , $r = \frac{3}{2}$ , $s = -\frac{1}{2}$ .

(f) B ——— A ——— X $\overrightarrow{AX} = -\frac{1}{2}\overrightarrow{AB}$ , $\overrightarrow{BX} = \frac{3}{2}\overrightarrow{BA}$ , $r = -\frac{1}{2}$ , $s = \frac{3}{2}$ .

6. (a) $\frac{1}{2}$
   (b) 2
   (c) $\overrightarrow{CX}$
   (d) $\overrightarrow{CA}$
   (e) $\frac{1}{2}$
   (f) $\frac{1}{2}$

---

## 11-2. Applications to Geometry.

This section has two main topics. The first is that vectors can be manipulated according to some of the usual rules of algebra. The second is that certain problems of elementary geometry can be solved by such manipulations.

Each of the examples is worked out as an isolated problem. No hint is given about a general approach to all of them. There is such a general approach which the teacher may want to discuss. Each problem can be solved by

(1) Choosing two directed line segments on non-parallel lines.

(2) Expressing each of the other directed line segments in terms of the ones originally selected.

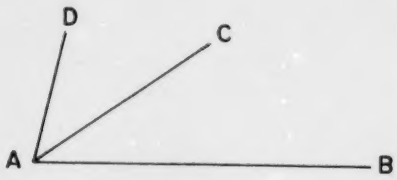
### Exercises 11-2. Answers.

1. (a) $\overrightarrow{DB} = \overrightarrow{DC} + \overrightarrow{DA}$, by Definition 11-1b and equivalent directed line segments.
   (b) $\overrightarrow{DB} = \overrightarrow{DC} + \overrightarrow{CB}$.
   (c) $\overrightarrow{DB} = \overrightarrow{DC} + \overrightarrow{CB} = \overrightarrow{AB} + (-\overrightarrow{BC}) = \overrightarrow{AB} - \overrightarrow{BC}$ [additive inverse].
   (d) $\overrightarrow{DB} = \overrightarrow{DA} + \overrightarrow{AB} = -\overrightarrow{AD} + \overrightarrow{AB}$.
   (e) $\overrightarrow{DB} = \overrightarrow{CB} + \overrightarrow{AB} = -\overrightarrow{BC} - \overrightarrow{BA}$.

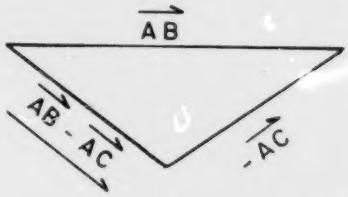
2. (a) The ray $\overrightarrow{AB}$ .
   (b) The segment $\overline{AB}$ .
   (c) The ray opposite to the ray $\overrightarrow{BA}$ .
   (d) The segment whose midpoint is A and which has B as an endpoint.

3. Hint: Note the development from the case where either r or s is zero and the other varies to the case where both are variable.
   (a) The line $\overleftrightarrow{AC}$ .
   (b) The line $\overleftrightarrow{AB}$ .
   (c) Any point between $\overleftrightarrow{AC}$ and $\overleftrightarrow{BX}$ where $\overleftrightarrow{BX} \parallel \overleftrightarrow{AC}$ .
   (d) Any point between $\overleftrightarrow{AB}$ and $\overleftrightarrow{CY}$ where $\overleftrightarrow{CY} \parallel \overleftrightarrow{AB}$ .
   (e) Any point inside the parallelogram ABCD where D is the intersection of $\overleftrightarrow{BX}$ and $\overleftrightarrow{CY}$ .
   (f) Any point on $\overleftrightarrow{BX}$ (line through B $\parallel$ $\overleftrightarrow{AC}$).
   (g) Any point on $\overleftrightarrow{CY}$ (line through C $\parallel$ $\overleftrightarrow{AB}$).
   (h) Any point on $\overleftrightarrow{BC}$ .
   (i) Any point on $\overleftrightarrow{BC'}$ where C' is on $\overleftrightarrow{CA}$ and A is the midpoint of segment $\overline{C'C}$ .
   (j) Any point on $\overleftrightarrow{PQ}$ where P is located on $\overleftrightarrow{AB}$ so that $\overrightarrow{AP} = 2\overrightarrow{AB}$ and Q is located on $\overleftrightarrow{AC}$ so that $\overrightarrow{AQ} = 3\overrightarrow{AC}$ .
   (k) Any point on $\overleftrightarrow{EF}$ where $\overrightarrow{AE} = \frac{4}{3}\overrightarrow{AB}$ and $\overrightarrow{AF} = \frac{8}{7}\overrightarrow{AC}$ (E on $\overrightarrow{AB}$ and F on $\overrightarrow{AC}$).
   (l) Any point on $\overleftrightarrow{GH}$ where $\overrightarrow{AG} = -\frac{c}{a}\overrightarrow{AB}$ and $\overrightarrow{AH} = -\frac{c}{b}\overrightarrow{AC}$.

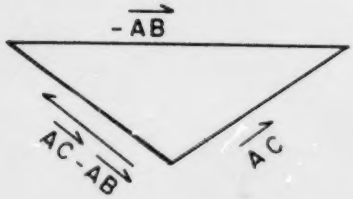
4.



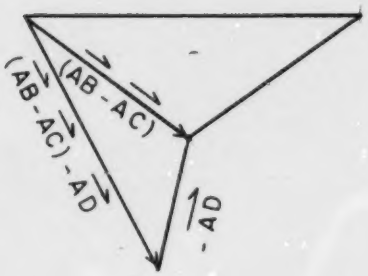

(a) $\overrightarrow{AB} - \overrightarrow{AC}$

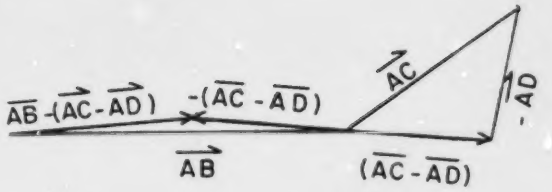



$\overrightarrow{AC} - \overrightarrow{AB}$



(b)




*5. Let $\overrightarrow{AG} = r\overrightarrow{AD}$ and $\overrightarrow{AE} = s\overrightarrow{AB}$. Then $\overrightarrow{HD} = (1-r)\overrightarrow{AD}$ and $\overrightarrow{EB} = (1-s)\overrightarrow{DC}$. Note that the opposite sides of a parallelogram are equal.

Let $\overrightarrow{V_1} = \overrightarrow{AH} + \ell\overrightarrow{HF}$ and $\overrightarrow{V_2} = \overrightarrow{AE} + m\overrightarrow{EG}$. We must show that these are values for $\ell$ and $m$ for which $\overrightarrow{V_1} = \overrightarrow{V_2}$ and that, for these values, either $V_1$ or $V_2$ equal some constant times $\overrightarrow{AC}$. $\overrightarrow{V_1} = \overrightarrow{V_2}$ implies

(1) $\overrightarrow{AH} + \ell(\overrightarrow{HD} + \overrightarrow{DF}) = \overrightarrow{AE} + m(\overrightarrow{EO} + \overrightarrow{OG})$.

Substituting for $\overrightarrow{AH}$, $\overrightarrow{AE}$ etc. in terms of $\overrightarrow{AD}$ and $\overrightarrow{DC}$ and collecting on $\overrightarrow{AD}$ and $\overrightarrow{DC}$ we obtain

(2) $(n + \ell - \ell n)\overrightarrow{AD} + \ell s\overrightarrow{DC} = (s + m - ms)\overrightarrow{DC} + mn\overrightarrow{AD}$

This equality (2) is satisfied if

(i) $n + \ell - \ell n = mn$ and (ii) $\ell s = s + m - ms$.

Solving (i) and (ii) for $\ell$ and $m$ in terms of $r$ and $s$ we obtain

(3) $\ell = \frac{n}{n + s - 1}$ ; $m = \frac{s}{n + s - 1}$ .

For these values of $\ell$ and $m$, $\overrightarrow{V_1} = \overrightarrow{V_2}$. Moreover

(4) $\overrightarrow{V_1} = \overrightarrow{AH} + \ell\overrightarrow{HF} = r\overrightarrow{AD} + \frac{r}{n + s - 1}[(\ell - r)\overrightarrow{AD} + s\overrightarrow{DC}]$

$= \frac{rs}{r + s - 1}(\overrightarrow{AD} + \overrightarrow{DC}) = \frac{rs}{r + s - 1} \cdot \overrightarrow{AC}$ .

Since $V_1$ equals a constant times $\overrightarrow{AC}$ the intersection of $\overrightarrow{HF}$ and $\overrightarrow{EG}$ lies on $\overrightarrow{AC}$. Q.E.D.

Question: What happens when $n + s = 1$?

6. (a) $\overrightarrow{OB} = \overrightarrow{OQ} + \overrightarrow{OP}$.

(b) $\overrightarrow{OC} = \overrightarrow{OQ} - \overrightarrow{OP}$ .

(c) $\overrightarrow{OD} = -\overrightarrow{OQ} - \overrightarrow{OP}$ .

(d) $\overrightarrow{OA} = -\overrightarrow{OQ} + \overrightarrow{OP}$ .

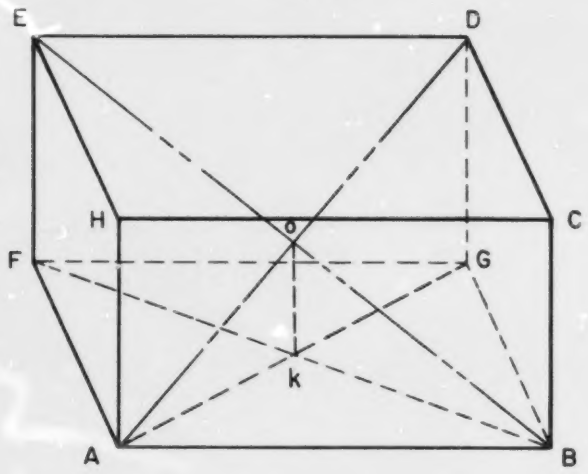
(e) $\overrightarrow{DB} = 2\overrightarrow{OQ} + 2\overrightarrow{OP}$ .

(f) $\overrightarrow{AC} = 2\ \overrightarrow{OQ} - 2\ \overrightarrow{OP}$ .

(g) $\overrightarrow{CA} = -2\ \overrightarrow{OQ} + 2\ \overrightarrow{OP}$ .

(h) $\overrightarrow{BD} = -2\ \overrightarrow{OQ} - 2\ \overrightarrow{OP}$ .

7.




Let O be the midpoint of AD, P the midpoint of BE, Q the midpoint of HG.

$\overrightarrow{AO} = \frac{1}{2}\overrightarrow{AD} = \frac{1}{2}(\overrightarrow{AG} + \overrightarrow{GD}) = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{BG} + \overrightarrow{GD}) = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AF} + \overrightarrow{AH})$

$\overrightarrow{AP} = \overrightarrow{AB} + \frac{1}{2}\overrightarrow{BE} = \overrightarrow{AB} + \frac{1}{2}(\overrightarrow{BF} + \overrightarrow{FE}) = \overrightarrow{AB} + \frac{1}{2}(\overrightarrow{BA} + \overrightarrow{AF} + \overrightarrow{FE})$

$= \overrightarrow{AB} - \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{AF} + \frac{1}{2}\overrightarrow{AH} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AF} + \overrightarrow{AH})$.

$\overrightarrow{AQ} = \overrightarrow{AH} + \frac{1}{2}\overrightarrow{HG} = \overrightarrow{AH} + \frac{1}{2}(\overrightarrow{HA} + \overrightarrow{AG}) = \overrightarrow{AH} + \frac{1}{2}(\overrightarrow{HA} + \overrightarrow{AB} + \overrightarrow{BG})$

$= \overrightarrow{AH} - \frac{1}{2}\overrightarrow{AH} + \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{AF} = \frac{1}{2}(\overrightarrow{AH} + \overrightarrow{AB} + \overrightarrow{AF})$

$\therefore\ \overrightarrow{AO} = \overrightarrow{AP} = \overrightarrow{AQ}$ and points O, P and Q coincide.

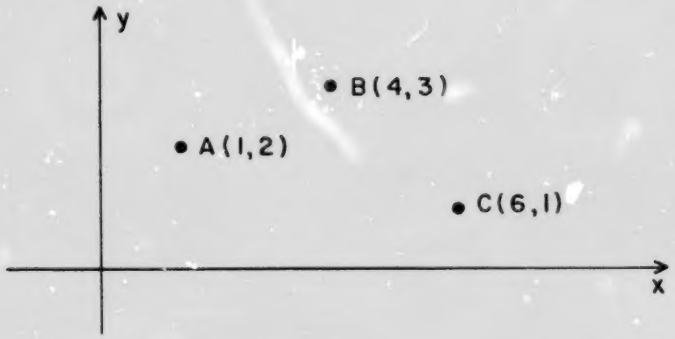
---

## 11-3. Vectors and Scalers; Components.

The main topic of this section is the algebra of vectors which is given in the component form [p,q] . The transition from coordinates (of points) to components (of vectors) is a little subtle. Once the change-over is made, the algebraic properties of vectors are easily established.

Exercises 11-3. Answers.

1.




(a) Let (a,b) be X .
Then $\overrightarrow{AB}$ is [(4 - 1) , (3 - 2)] .
Then $\overrightarrow{CX}$ is [(a - 6) , (b - 1)] .
Since $\overrightarrow{AB} \doteq \overrightarrow{CX}$ ,

$$a - 6 = 4 - 1 \quad , \quad b - 1 = 3 - 2$$
$$a = 9 \qquad\qquad b = 2 .$$

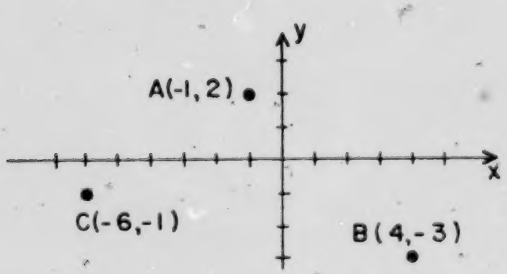
The coordinates of X are (9,2) .

(b) $a - 1 = 4 - 6 \quad , \quad b - 2 = 3 - 1$
$a = -1 \qquad\qquad b = 4$
X(-1,4)

(c) $1 - a = 4 - 6 \quad , \quad 2 - b = 3 - 1$
$a = 1 \qquad\qquad b = 0$
X(1,0)

(d) $1 - a = 6 - 4 \quad , \quad 2 - b = 1 - 3$
$a = -1 \qquad\qquad b = 4$
X(-1,4)

2. (a) $4 - (-1) = a - (-6) \quad , \quad -3 - 2 = b - (\quad)$
$a = -1 \qquad\qquad b = -6$
X(-1,-6)



(b) $a - (-1) = 4 - (-6)$ , $b - 2 = -3 - (-1)$

$a = 9$ , $b = 0$

$X(9,0)$

(c) $-1 - a = 4 - (-6)$ , $2 - b = -3 - (-1)$

$a = -11$ , $b = 4$

$X(-11,4)$

(d) $X(9,0)$ .

3. (a) $[3,2] + [4,1] = [(3 + 4) + (2 + 1)] = [7,5]$ , by Theorem 11-3b.

(b) $[1,-1]$ .

(c) $4[5,6] = [4 \cdot 5 , 4 \cdot 6] = [20,24]$ , by Theorem 11-3c.

(d) $[-20,-24]$, by Theorem 11-3c.

(e) $[-5,-6]$ , by Corollary of Theorem 11-3c.

(f) $[-5,-6]$ .

(g) $3[4,1] + 2[-1,3] = [12,3] + [-2,6] = [12 + (-2) , 3 + 6]$
$= [10,9]$ .

(h) $[14,-3]$ .

4. (a) $x[3,-1] + y[3,1] = [5,6]$

$[3x,-x] + [3y,y] = [5,6]$

$[(3x + 3y) , (-x + y)] = [5,6]$

$$\begin{cases} 3x + 3y = 5 \\ -x + y = 6 \end{cases}$$

The solution set of the system is $\{(\frac{-13}{6},\frac{23}{6})\}$ .

That is, $x = \frac{-13}{6}$ and $y = \frac{23}{6}$ .

(b) The resulting system is,

$$\begin{cases} 3x + 2y = 1 \\ 2x + 3y = 2 \end{cases}$$ whose solution set is $\{(-\frac{1}{5},\frac{4}{5})\}$ .

That is, $x = -\frac{1}{5}$ and $y = \frac{4}{5}$ .

(c) $x = \frac{27}{13}$ and $y = \frac{8}{13}$ .

(d) The solution set of the system

$$\begin{cases} 3x + 6y = -3 \\ 2x + 4y = -2 \end{cases}$$ is $\{(a, \frac{-a - 1}{2})\}$ for all real $a$.

For instance, one element of the solution set is $(3, \frac{-3 - 1}{2})$, or $(3,-2)$. Ask students to find other pairs of numbers which belong to the solution set. There will be an infinite number of such pairs.

5. (a) $[3,1] = a[1,0] + b[0,1]$

$[3,1] = [a,0] + [0,b]$

$[3,1] = [(a + 0), (0 + b)]$

$a + 0 = 3$ and $0 + b = 1$

$a = 3$ and $b = 1$

(b) $a = 1$ and $b = -3$.

(c) $\vec{i} = a[-3,1] + b[1,-3]$

$[1,0] = a[-3,1] + b[1,-3]$

$[1,0] = [-3a,a] + [b,-3b]$

$[1,0] = [(-3a + b), (a - 3b)]$.

Hence a and b satisfy

$$\begin{cases} -3a + b = 1 \\ a - 3b = 0 \end{cases}.$$

We conclude that $a = -\frac{3}{8}$ and $b = -\frac{1}{8}$.

(d) $\vec{j} = a[-3,1] + b[1,-3]$

$[0,1] = a[-3,1] + b[1,-3]$.

Hence a and b satisfy

$$\begin{cases} -3a + b = 0 \\ a - 3b = 1 \end{cases}.$$

We conclude that $a = -\frac{1}{8}$ and $b = -\frac{3}{8}$.

6. $3\vec{i} - 2\vec{j} = a(3\vec{i} + 4\vec{j}) + b(4\vec{i} + 3\vec{j})$
$3\vec{i} + (-2)\vec{j} = (3a\vec{i} + 4a\vec{j}) + (4b\vec{i} + 3b\vec{j})$
$3\vec{i} + (-2)\vec{j} = (3a + 4b)\vec{i} + (4a + 3b)\vec{j}$ .
Hence a and b satisfy
$$\begin{cases} 3a + 4b = 3 \\ 4a + 3b = -2 . \end{cases}$$
We conclude that $a = -\frac{17}{7}$ and $b = \frac{18}{7}$ .

---

## 11-4. Inner Product.

The system of vectors before the inner product is introduced is not adequate to handle all of geometry. Only a few problems relating to angles and distance can be covered. The introduction of the inner product enriches vector algebra to the point that it is capable of being a completely adequate substitute for Euclidean Geometry.

The student is not likely to see these implications of the introduction of inner product. He should only be expected to compute them and to use them in the simple applications indicated.

### Exercises 11-4. Answers.

1. Given $\vec{i} = [1,0]$ and $\vec{j} = [0, 1]$ .
   (a) $\overline{X} \cdot \overline{Y} = [1,0] \cdot [0,1] = 1 \cdot 0 + 0 \cdot 1 = 0$ .
   (b) $[1,0] \cdot [1,0] = 1 \cdot 1 + 0 \cdot 0 = 1$ .
   (c) $[0,1]\ [1,0] = 0$ .
   (d) 1 .
   (e) 0 .
   (f) -7 .
   (g) -7 .
   (h) ac + bd .
   (i) $4a^2 + 4b^2$ .
   (j) $s\,a^2 + s\,b^2$ .

2. $\vec{X} \cdot \vec{Y} = |\vec{X}||\vec{Y}|\cos\theta$ .

(a) $2 \cdot 3 \cos\theta = 0$ ; therefore, $\theta = 90^\circ$ .

(b) $81.4^\circ$ .

(c) $109.5^\circ$ .

(d) $60^\circ$ .

(e) $33.6^\circ$ .

(f) $180^\circ$ .

3. If $\vec{Y} \perp \vec{X}$ then $\vec{X} \cdot \vec{Y} = 0$ .

(a) $[3,4][a,4] = 0$

$3a + 16 = 0$ ; $a = -\frac{16}{3}$ .

(b) $\frac{16}{3}$ .

(c) $-3$ .

(d) $4$ .

4. (a) $\sqrt{1^2 + 0^2}\sqrt{0^2 + 1^2}\cos\theta = 0$

$\cos\theta = 0$ ; $\theta = 90^\circ$ .

(b) $0$ .

(c) $90^\circ$ .

(d) $0$ .

(e) $90^\circ$ .

(f) $107.6^\circ$ .

(g) $107.6^\circ$ .

(h) $\cos\theta = \dfrac{ac + bd}{\sqrt{(a^2 + b^2)(c^2 + d^2)}}$ .

(i) $\sqrt{a^2 + b^2}\sqrt{16(a^2 + b^2)}\cos\theta = 4(a^2 + b^2)$

$\cos\theta = 1$

$\theta = 0^\circ$ .

(j) $0^\circ$ .

5. Note that $(c\vec{i} + d\vec{j}) \cdot (-d\vec{i} + c\vec{j}) = -cd + cd = 0$ .
Therefore since $c^2 + d^2 \neq 0$ , $c\vec{i} + d\vec{j}$ is perpendicular to $-d\vec{i} + c\vec{j}$ . A non-zero vector is perpendicular to one of these if and only if it is parallel to the other.

6. (a) Component of $\vec{Y}$ in the direction of $\vec{X}$ is $|\vec{Y}| \cos\theta$.

$|\vec{Y}| = \sqrt{3^2 + 4^2} = 5$. To find $\theta$ we note two expressions of $\vec{X} \cdot \vec{Y}$. (i) $\vec{X} \cdot \vec{Y} = |\vec{X}| \cdot |\vec{Y}| \cos\theta$ and

(ii) $\vec{X} \cdot \vec{Y} = x_1y_1 + x_2y_2$ where $\vec{X} = x_1i + x_2j$

and $\vec{Y} = y_1i + y_2j$.

From (i) and (ii) we have $x_1y_1 + x_2y_2 = |\vec{X}||\vec{Y}| \cos\theta$

$$1 \cdot 3 + 0 \cdot 4 = \sqrt{1^2 + 0^2} \cdot \sqrt{3^2 + 4^2} \cos\theta$$

$$3 = 5 \cos\theta \longrightarrow \cos\theta = \frac{3}{5}$$

Desired component $= 5 \cdot \frac{3}{5} = 3$.

(b) Using same plan as in part (a) we obtain $4 = 5 \cos\theta$.

$\therefore$ Component of $\vec{Y}$ in direction $\vec{X} = 4$.

(c) $3 \cdot 1 + 4 \cdot 0 = 5 \cdot 1 \cos\theta$. $\quad \cos\theta = \frac{3}{5}$

$|\vec{Y}| \cos\theta = 1 \cdot \frac{3}{5} = \frac{3}{5}$.

(d) $3 \cdot 0 + 4 \cdot 1 = 5 \cdot 1 \cos\theta$. $\quad \cos\theta = \frac{4}{5}$.

$|\vec{Y}| \cos\theta = \frac{4}{5}$

(e) $3 \cdot 3 + 4 \cdot 4 = 5 \cdot 5 \cos\theta$. $\quad \cos\theta = 1$

component $= 5 \cdot 1 = 5$.

(f) $15 + 8 = 5 \cdot \sqrt{29} \cos\theta$. $\quad \cos\theta = \frac{23}{5\sqrt{29}}$

$$\vec{Y} \cos\theta = \sqrt{29} \cdot \frac{23}{5\sqrt{29}} = \frac{23}{5} = 4.6$$

(g) $3a + 4b = 5 \cdot \sqrt{a^2 + b^2} \cdot \cos\theta$

$$\frac{3a + 4b}{5\sqrt{a^2 + b^2}} = \cos\theta$$

$|\vec{Y}| \cos\theta = \frac{3a + 4b}{5}$ = desired component.

(h) $pa + qb = \sqrt{p^2 + q^2} \cdot \sqrt{a^2 + b^2} \cos \theta$

Desired component $= \sqrt{a^2 + b^2} \cos \theta = \dfrac{pa + qb}{\sqrt{p^2 + q^2}}$

---

11-5. Applications of Vectors in Physics.

The main topic of this section is the use of vectors in solving certain problems of physics. The student does not have to know much in the way of physics to handle the material, but there are a few bits of information which are taken for granted in the problems (for instance, that the direction of a force transmitted by a cord must be along the line of the cord). Primarily the student should come to this work knowing about addition of vectors, scalar multiplication and inner products. He should see how this knowledge can help him to learn substantial amounts of physics easily. For instance, forces in equilibrium can be discussed readily in vector language.

Two extreme points of view should be avoided.

(1) The student could get the impression that his knowledge of vectors makes him an expert physicist. This is not so. He needs to learn a little physics as well as vector algebra to solve these problems.

(2) The student could get the impression that in spite of his knowledge of vectors he is unable to solve the simple problems given here without a lot of supplementary study of physics. This is not so. He is given a few observations on forces, resultant of forces, forces in equilibrium, work, velocity. These should not be made to appear so formidable as to discourage him.

Exercises 11-5a. Answers.

1. $5\sqrt{2}$ lb.

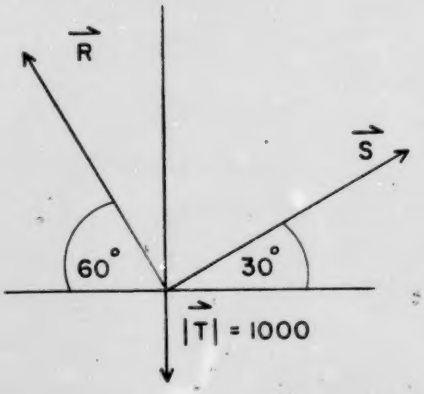
2. $\vec{R} = (|\vec{R}|\cos 120^\circ , |\vec{R}|\sin 120^\circ)$

$= (-\frac{1}{2}|\vec{R}| , \frac{|\vec{R}|\sqrt{3}}{2})$ .

$\vec{S} = (|\vec{S}|\cos 30^\circ , |\vec{S}|\sin 30^\circ)$

$= (\frac{|\vec{S}|\sqrt{3}}{2} , \frac{1}{2}|\vec{S}|)$ .

$\vec{T} = (0, -1000)$ .




$\vec{R} + \vec{S} + \vec{T} = \vec{0}$ .

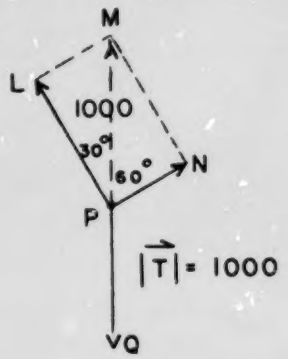
$(-\frac{1}{2}|\vec{R}| + \frac{|\vec{S}|\sqrt{3}}{2} , \frac{|\vec{R}|\sqrt{3}}{2} + \frac{1}{2}|\vec{S}| - 1000) = (0,0)$ .

$|\vec{R}| = 500\sqrt{3} \approx 866$ .

$|\vec{S}| = 500$ .

The force of wire AC on C is approximately 866 pounds; the force of wire BC on C is 500 pounds; for equally strong wires, CW is more likely to break since the greatest force is on it and BC is least likely to break.

An alternate solution can be gained using "free" vectors, right triangles, and the resultant of $\vec{R}$ and $\vec{S}$ as shown in the sketch. Using the parallelogram law for the addition of the vectors, $\vec{PM}$ must be the hypotenuse of a $30^\circ$ - $60^\circ$ right triangle and have a length of 1000 units. Hence, $\vec{PN}$ which lies opposite the $30^\circ$ angle has a length of 500 units. Similarly in right triangle LMP, $\vec{LP}$ lies opposite the $60^\circ$ angle; it has a length of $500\sqrt{3}$ units.



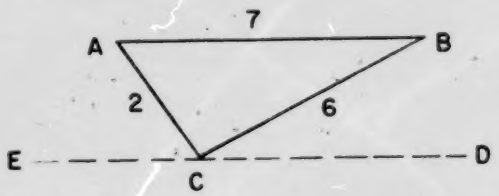

3. Force in AC is $10000/\sqrt{3} \approx 5770$ .
Force in BC is $5000/\sqrt{3} \approx 2885$ .
Force in CW is 5000 pounds.

4. $\vec{OP} = (|\vec{OP}| \cos 23^\circ, |\vec{OP}| \sin 23^\circ)$
$\vec{OQ} = (|\vec{OQ}| \cos 113^\circ, |\vec{OQ}| \sin 113^\circ)$ $\vec{OW} = (0,-300)$
$\vec{OP} + \vec{OQ} + \vec{OW} = 0$ ie. $(|\vec{OP}| \cos 23^\circ + |\vec{OQ}| \cos 113^\circ,$
$|\vec{OP}| \sin 23^\circ + |\vec{OQ}| \sin 113^\circ - 300) = (0,0)$
Solving $|\vec{OP}| \approx 117$ and $|\vec{OQ}| \approx 276$ .

5. From the Law of Cosines,

$$\cos B = \frac{7^2 + 6^2 - 2^2}{(2)(7)(6)}$$

$$= \frac{27}{28} \approx 0.971 .$$

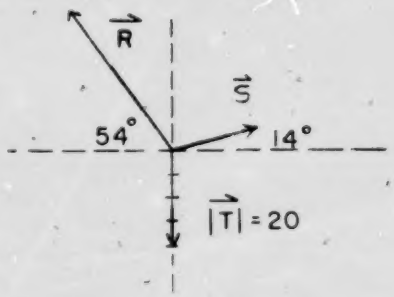



Angle B $\approx 14^\circ$ = angle BCD .

Also, $\cos C = \frac{2^2 + 6^2 - 7^2}{(2)(2)(6)} = -\frac{3}{8} \approx -0.3750$, and $C \approx 112^\circ$ .

Angle ACE $\approx 180^\circ - (14^\circ + 112^\circ) = 54^\circ$

Forming a vector diagram in a coordinate system with a vector unit of 1 pound of force, the components of the vectors are:



$\vec{R} = (|\vec{R}|\cos 54^\circ , |\vec{R}|\sin 54^\circ) \approx (-0.588\ |\vec{R}| , 0.809\ |\vec{R}|)$ ;

$\vec{S} = (|\vec{S}|\cos 14^\circ , |\vec{S}|\sin 14^\circ) \approx (0.970\ |\vec{S}| , 0.242\ |\vec{S}|)$ ;

$\vec{T} = (0,-20)$ .

Since $\vec{R} + \vec{S} + \vec{T} = \vec{0}$ , adding the left member vectors gives the equal vectors

$$(-0.588\ |\vec{R}| + 0.970\ |\vec{S}| , 0.809\ |\vec{R}| + 0.242\ |\vec{S}| - 20) = (0,0) .$$

Equating corresponding components, we have

$$-0.588\ |\vec{R}| + 0.970\ |\vec{S}| = 0 ,$$

and $$0.809\ |\vec{R}| + 0.242\ |\vec{S}| = 20 .$$

Solving these equations simultaneously, we have

$$|\vec{R}| = \frac{0.970}{0.588}\ |\vec{S}| .$$

$$0.809\ (\frac{0.970}{0.588})\ |\vec{S}| + 0.242\ |\vec{S}| = 20 .$$

$$(1.334 + 0.242)\ |\vec{S}| = 20 .$$

$$|\vec{S}| = \frac{20}{1.576} \approx 12.7 .$$

$$|\vec{R}| = \frac{0.970}{0.588}\ (12.7) \approx 21.0 .$$

The force on wire AC is approximately 21 pounds; on wire BC , approximately 12.7 pounds. Wire AC is the one which is most likely to break.

6. The force on wire BC at C is $500\sqrt{3} \approx 866$ pounds; on AC at C , it is 1000 pounds; and on CW at C , it is 500 pounds.

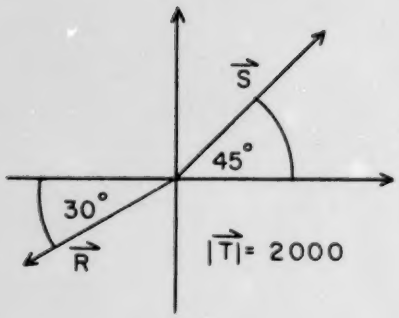
7. $\vec{R} = (|\vec{R}|\cos 210^\circ , |\vec{R}|\sin 210^\circ)$

$= (-\frac{\sqrt{3}}{2}|\vec{R}| , -\frac{1}{2}|\vec{R}|)$ .

$\vec{S} = (|\vec{S}|\cos 45^\circ , |\vec{S}|\sin 45^\circ)$

$= (\frac{1}{\sqrt{2}}|\vec{S}| , \frac{1}{\sqrt{2}}|\vec{S}|)$ .

$\vec{T} = (0, -2000)$ .




Since $\vec{R} + \vec{S} + \vec{T} = \vec{0}$ , addition gives two equal vectors; thus

$$(-\frac{\sqrt{3}}{2}|\vec{R}| + \frac{1}{\sqrt{2}}|\vec{S}| , -\frac{1}{2}|\vec{R}| + \frac{1}{\sqrt{2}}|\vec{R}| - 2000) = (0,0) .$$

Equating corresponding components gives the following pair of simultaneous equations:

$$-\frac{\sqrt{3}}{2}|\vec{R}| + \frac{1}{\sqrt{2}}|\vec{S}| = 0 ,$$

$$-\frac{1}{2}|\vec{R}| + \frac{1}{\sqrt{2}}|\vec{S}| - 2000 = 0 .$$

In the first equation,

$$|\vec{R}| = \frac{2}{\sqrt{6}}|\vec{S}| .$$

Using this value in the second equation, we obtain

$$(-\frac{1}{\sqrt{6}} + \frac{1}{\sqrt{2}})|\vec{S}| = 2000 .$$

Hence, $|\vec{S}| = \frac{\sqrt{12}\,(2000)}{\sqrt{6} - \sqrt{2}} \approx 6750$

$$|\vec{R}| = \frac{\sqrt{2}\,(4000)}{\sqrt{6} - \sqrt{2}} \approx 5465$$

8. The vectors are placed in a coordinate system using 1,000 pounds of force as a convenient vector unit. The vector components are as follows:

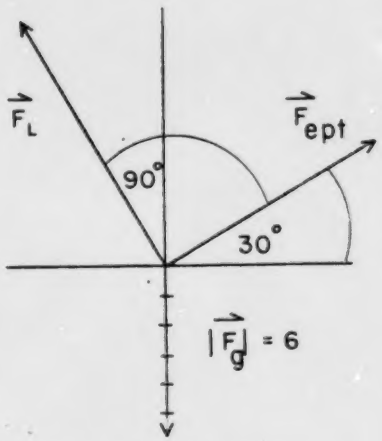
$$\overrightarrow{F_{ept}} = [|\overrightarrow{F_{ept}}|\cos 30^\circ \ , \ |\overrightarrow{F_{ept}}|\sin 30^\circ]$$

$$\overrightarrow{F_{ept}} = \left[\frac{|\overrightarrow{F_{ept}}|\sqrt{3}}{2} \ , \ \frac{|\overrightarrow{F_{ept}}|}{2}\right] ;$$

$$\overrightarrow{F_L} = [|\overrightarrow{F_L}|\cos 120^\circ \ , \ |\overrightarrow{F_L}|\sin 120^\circ]$$

$$= \left[\frac{-|\overrightarrow{F_L}|}{2} \ , \ \frac{|\overrightarrow{F_L}|\sqrt{3}}{2}\right] ;$$

$$\overrightarrow{F_g} = (0,-6) .$$




Since the airplane is moving in a straight line at constant speed,

$$\overrightarrow{F_{ept}} + \overrightarrow{F_L} + \overrightarrow{F_g} = \overrightarrow{0} .$$

Adding the vectors in the left member we obtain

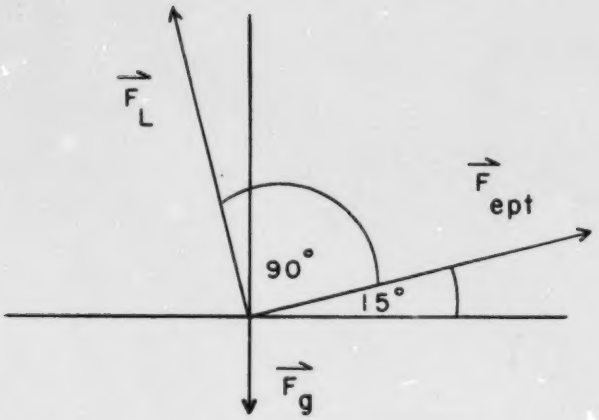
$$\left[\frac{|\overrightarrow{F_{ept}}|\sqrt{3}}{2} - \frac{|\overrightarrow{F_L}|}{2} + 0 \ , \ \frac{|\overrightarrow{F_{ept}}|}{2} + \frac{|\overrightarrow{F_L}|\sqrt{3}}{2} - 6\right] = (0,0) .$$

Equating corresponding components and solving the equations simultaneously, we obtain

$$|\overrightarrow{F_{ept}}| = 3.000 \ , \text{ and } \ |\overrightarrow{F_L}| = 3\sqrt{3} \approx 5.196 .$$

Hence, the effective propeller thrust is 3000 pounds and the lift force is approximately 5196 pounds.

9.




$\vec{F}_L = (|\vec{F}_L|\cos 105^\circ , |\vec{F}_L|\sin 105^\circ)$

$= (-0.26\ |\vec{F}_L| , 0.97\ |\vec{F}_L|)$ .

$\vec{F}_{ept} = (|\vec{F}_{ept}|\cos 15^\circ , |\vec{F}_{ept}|\sin 15^\circ)$

$= (0.97\ |\vec{F}_{ept}| , 0.26\ |\vec{F}_{ept}|)$ .

$\vec{F}_g = (0,-10,000)$ .

Since $\vec{F}_L + \vec{F}_{ept} + \vec{F}_g = \vec{0}$ , we have the two simultaneous equations:

(1) $-0.26\ |\vec{F}_L| + 0.97\ |\vec{F}_{ept}| = 0$ ,

(2) $0.97\ |\vec{F}_L| + 0.26\ |\vec{F}_{ept}| = 10,000$ .

Solving this system of equations, we get

$|\vec{F}_L| = 3.7\ |\vec{F}_{ept}|$ ,

and $|\vec{F}_{ept}| = 2,600$ pounds .

$|\vec{F}_L| = 9,500$ pounds .

10. $\vec{F_L} = (|\vec{F_L}|\cos 100^\circ , |\vec{F_L}|\sin 100^\circ) \approx (-0.174\ |\vec{F_L}|, 0.985\ |\vec{F_L}|)$ .

$\vec{F_d} = (|\vec{F_d}|\cos 10^\circ , |\vec{F_d}|\sin 10^\circ) \approx (0.985\ |\vec{F_d}| , 0.174\ |\vec{F_d}|)$ .

$\vec{F_g} = (0,-500)$ .

$-0.174\ |\vec{F_L}| + 0.985\ |\vec{F_d}| = 0$ .

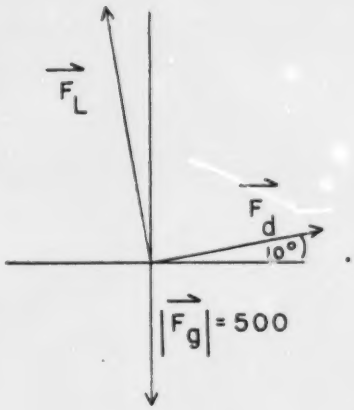
$0.985\ |\vec{F_L}| + 0.174\ |\vec{F_d}| - 500 = 0$ .

$|\vec{F_L}| = \frac{0.985}{0.174}\ |\vec{F_d}|$ .

$(0.985)\ (\frac{0.985}{0.174}\ |\vec{F_d}|) + 0.174\ |\vec{F_d}| = 500$

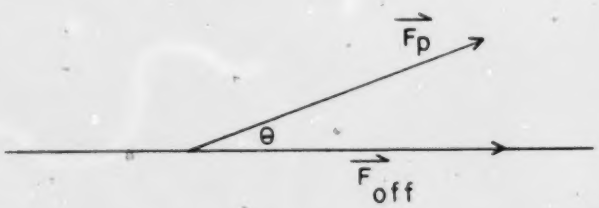
$|\vec{F_d}| = \frac{500}{5.56 + 0.174} \approx 87.3$ .

$|\vec{F_L}| \approx \frac{0.985}{0.174}\ (87.3) \approx 494$ .




Exercises 11-5b. Answers.

1.




$\vec{F_{ebb}} = \vec{F_p} \cos \theta$

$W = d \cdot \vec{F_{ebb}}$

(a) $\vec{F_{ebb}} = 10 \cos 10^\circ = 10 \cdot .985 = 9.85$ lb.

Work $= d \cdot \vec{F_{ebb}} = 10 \times 9.85 = 98.5$ ft. lb.

(b) $W = 100 \cdot 10 \cdot \cos 20^\circ = 1000 \cdot 940 = 940$ ft. lb.

(c) $W = 8660$ ft. lb.

(d) $W = d \cdot 10 \cos 10^\circ$

$1000 = d \cdot 10 \cdot .985$

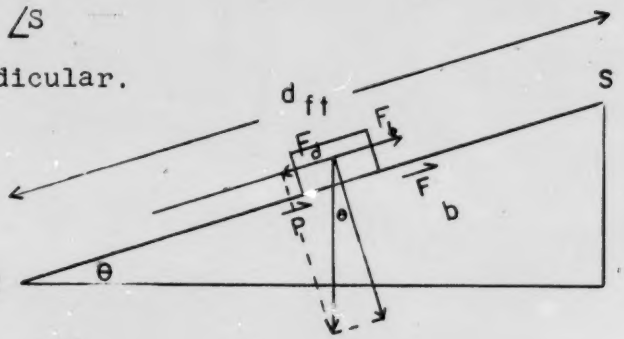
$d = \frac{1000}{9.85} = 101.5$ ft.

(e) $d = \frac{1000}{100 \cdot .940} = \frac{100}{94} = 10.6$ ft.

(f) $d = \frac{1000}{100 \cdot \cos 0^c} = 10$ ft.

(g) $d = \frac{1000}{100 \cos 89^\circ} = \frac{1000}{100 \cdot .0175} = \frac{1000}{1.75} = 571.4$ ft.

2. $\angle$ between P and $F_b$ = $\theta$ since sides of $\angle$S are mutually perpendicular.




(a) $|\vec{F_d}| = |\vec{P}| \cos(\frac{\pi}{2} - \theta)$ $\quad |\vec{F_d}| = |\vec{P}| \sin \theta \quad \vec{F_b} = -\vec{F_d}$

$W = \vec{F_b} \cdot \vec{d}$

$W = d \vec{P} \sin \theta$ (Note that this is equivalent to lift-P in a vertical direction from R to S .)

$W = 10 \cdot 10 \cdot \sin 10^\circ$

$W = 100 \cdot .174 = 17.4$ ft. lb.

(b) $W = 342$ ft. lb.

(c) W = 500 ft. lb.

(d) $d = \frac{W}{P \sin \theta}$

$d = 575$ ft.

(e) $d = 292$ ft.

(f) $d = 571$ ft.

(g) $d = 10$ ft.

Exercises 11-5c. Answers.

1. $\approx 1.8$ miles .

2. From fig. (a) we determine the angle which the path of the boat makes with the shore line ($\angle \alpha$) and the speed of the boat along OQ. Let length of OQ = d.

$d^2 = 1.3^2 + 0.5^2$

$d = \sqrt{1.94} = 1.39$ = distance traveled.

The boat covers the distance d in 25 minutes. Hence if s is the speed of the boat along OQ

$s \cdot \frac{5}{12} = 1.39$

$s = 3.34$

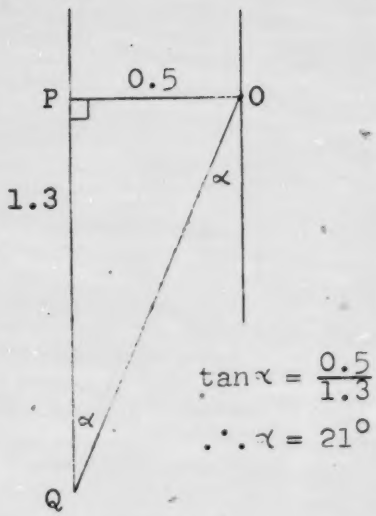



Figure (a)

Figure (b) is our force diagram. We have

$\overrightarrow{OT} + \overrightarrow{TR} = \overrightarrow{OR}$

$|\overrightarrow{OT}| = 4 \qquad |\overrightarrow{OR}| = 3.34$

and $\angle TOR = 21^\circ$ .

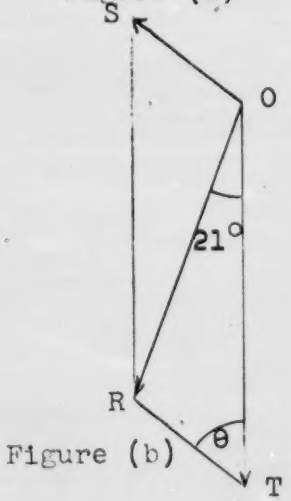



Figure (b)

By the Cosine Law

$$|\overline{TR}|^2 = 4^2 + (3.34)^2 - 2 \cdot 4 \cdot 3.34 \cos 21^\circ$$

$$|\overline{TR}| \approx 1.52$$

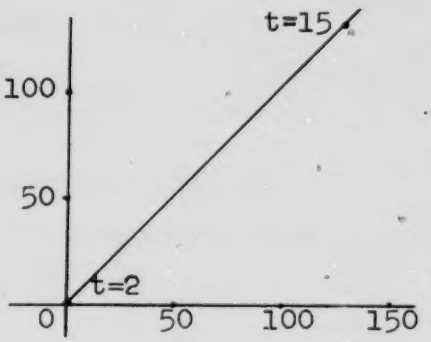
By the Sine Law applied to $\Delta$ RTO

$$\frac{3.34}{\sin \theta} = \frac{1.52}{\sin 21^\circ} . \qquad \theta \approx 51^\circ$$

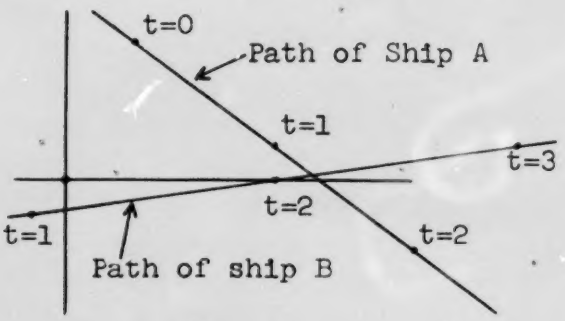
3. $\sqrt{37} \approx 6.08$ miles per hour.

4. Since the velocity is constant, in one second the body will reach the point $(2,1.5)$ . Thus, the velocity vector is $2\vec{i} + 1.5\vec{j}$ . The velocity of the body is 200 feet per second to the right, and 150 feet per second upward. Its speed is 250 feet per second.

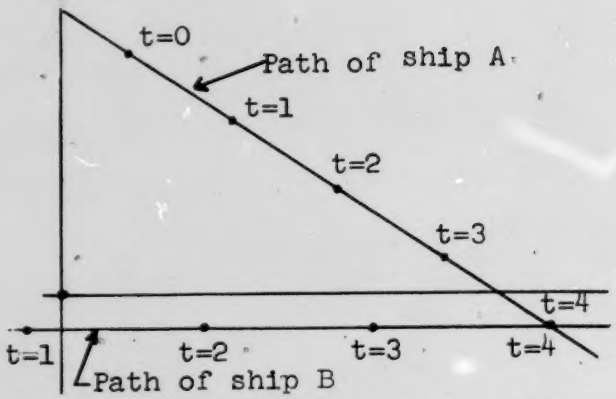
5. At $t = 15$ the body is at the point $(130,131)$ . Thus, it has moved 1300 miles to the right, and 1310 miles upward.




6. Since ship B does not cross the wake of ship A until after $t = 2$ , the ships will not collide.



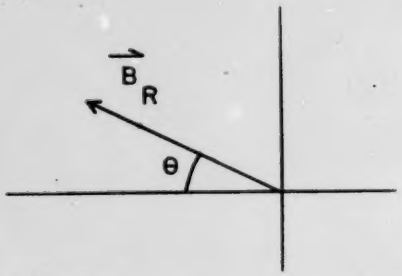
7. Since both ships are at the point (14,-1) when $t = 4$, the ships will collide.




8. We compute the displacements that would result from one hour of travel. Thus,

$$\vec{R_L} = -4\vec{j},$$

$$\vec{B_R} = -3(\cos\theta)\vec{i} + 3(\sin\theta)\vec{j}.$$

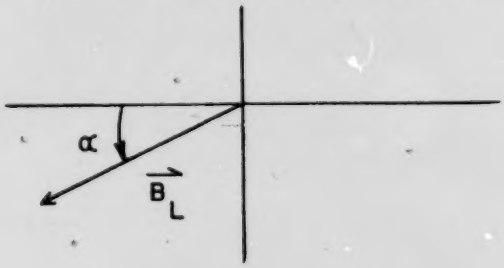



Consequently,

$$\vec{B_L} = \vec{R_L} + \vec{B_R}$$

$$= (-3\cos\theta)\vec{i} + (3\sin\theta - 4)\vec{j}.$$

The scalar components of $\vec{B_L}$ are both negative. This means that the boat will actually be carried downstream. The situation is illustrated by the diagram below.




In order to drift downstream as little as possible $\theta$ must be determined so that $\tan\alpha$ is minimum:

$$\tan\alpha = \frac{-3\sin\theta + 4}{3\cos\theta}.$$

This problem can be handled easily by using calculus. However, by making use of a table or graph we can obtain an approximate solution without using calculus. Thus, the smallest value of $\tan \alpha$ occurs for

$$\sin \theta = \frac{3}{4},$$

$$\theta \approx 49^\circ .$$

For this value of $\theta$,

$$\vec{B_L} = -3(.66)\vec{i} + (3 \times \tfrac{3}{4} - 4)\vec{j}$$

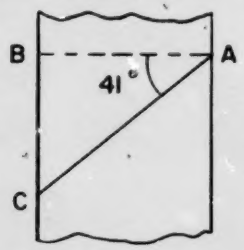
$$= -1.98\vec{i} - 1.75\vec{j} .$$

The corresponding value of $\alpha$ is given by

$$\tan \alpha = \frac{-3\left(\frac{3}{4}\right) + 4}{\frac{3\sqrt{7}}{4}} = \frac{7}{3\sqrt{7}} = 0.88 .$$

$$\approx 41^\circ .$$

Travelling in this direction, the boat will land at C. Thus




$AB = \frac{1}{2}$ mile,

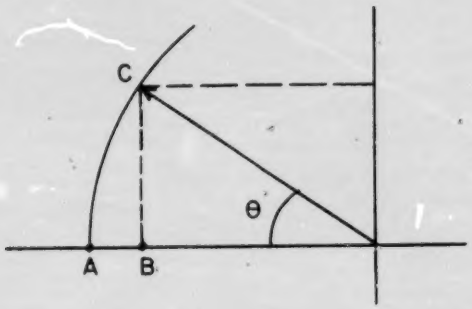
$BC = \frac{1}{2} \tan \alpha = \frac{1}{2} \times (.88) = 0.44$ miles.

Therefore, the boat must be carried <u>at least</u> 0.44 miles downstream. Another way of saying the same thing is that C is the <u>farthest</u> point <u>upstream</u> at which the man can land the boat.

The intuitive meaning of this problem is quite subtle. Let us consider the effect of different values of $\theta$. Evidently if $\theta < 0$, then the man is using a component of his rowing <u>to help the current sweep him downstream</u>. This is the very opposite of what he wishes to do.

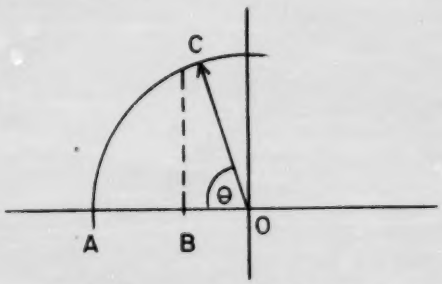
Hence, a wise choice of $\theta$ requires that $0 \leq \theta \leq \frac{\pi}{2}$.

It might seem sensible to head straight for the opposite shore; i.e., to choose $\theta = 0$. Let us examine this possibility carefully.




If $\theta$ is chosen so that it is about $49^\circ$, then the man will have sacrificed a component $\overrightarrow{BA}$ which would carry him to the opposite shore, but he will have gained a much larger component $\overrightarrow{BC}$ which is keeping him from being swept downstream. For $\theta \approx 49^\circ$ he is crossing almost as fast as he would be for $\theta = 0^\circ$, but he is not being swept downstream so rapidly. It is a good bargain.

What would happen if the man sacrificed even <u>more</u> of the crossing component in order to gain a larger component working against the current? Suppose he chooses $\theta = 70^\circ$. In doing so he sacrifices a crossing component of $\overrightarrow{BA}$ in order to gain the component $\overrightarrow{BC}$ which opposes the current. The price is too great, however.



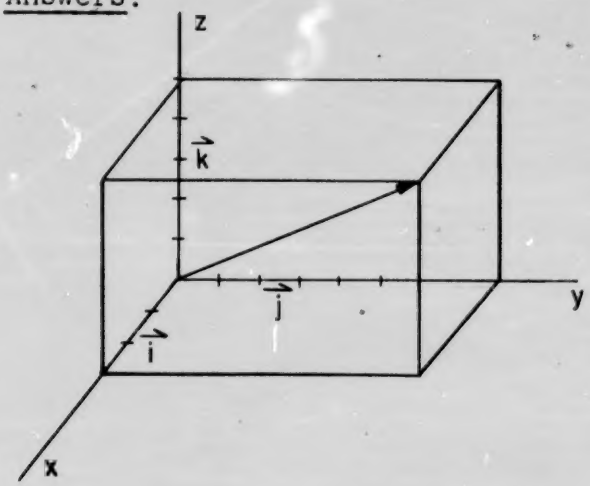

Even though the man is not being swept downstream <u>so rapidly</u>, he will actually be swept farther downstream. This is true because

the crossing component $\overrightarrow{OB}$ is now very small; consequently, it takes him a long time to cross. During this time, he is swept, slowly but surely, a long way down the stream.

Finding the optimum value of $\theta$ is, therefore, a matter of compromise; it is motivated by a desire to oppose the current as much as possible, without slowing progress toward the opposite shore more than a little.

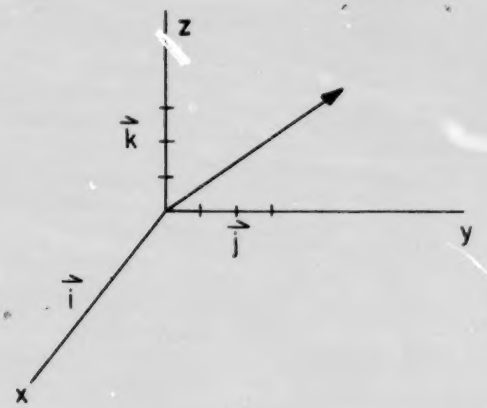
Exercises 11-5d. Answers.

1. (a)




$3\vec{i} + 8\vec{j} + 5\vec{k}$

(b)




$3\vec{j} + 3\vec{k}$

Since the graphs for each of the remaining parts of this problem are similar to (a) and (b), they have been omitted.

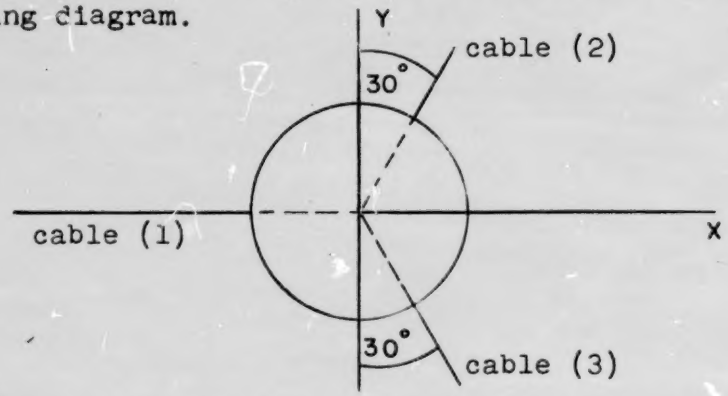
2. (a) 16 .
   (b) 10 .
   (c) 0 .
   (d) 0 .
   (e) 0 .
3. (a) $\frac{16}{3\sqrt{29}}$ .

   (b) $\frac{10}{\sqrt{29}\sqrt{12}}$ .

   (c) 0 .
   (d) 0 .
   (e) 0 .
4. 0 .

*5. We shall give two solutions to this problem.

<u>First solution</u>: Let us first find vectors having the directions of the suspending cords. By orienting axes appropriately we obtain the top view represented in the following diagram.




Let $\vec{A}$ be a vector which is parallel to cable (1) . Then the vector $-\vec{i}$ makes an angle of $30^\circ$ with $\vec{A}$ , an angle of $60^\circ$ with $\vec{k}$ , and an angle of $90^\circ$ with $\vec{j}$ .

Let $\vec{A} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$ .

If we "dot" $\vec{j}$ into both sides of this equation, we get

$$\vec{j} \cdot \vec{A} = \vec{j} \cdot (a_x\vec{i} + a_y\vec{i} + a_z\vec{k})$$

$$= a_x (\vec{j} \cdot \vec{i}) + a_y (\vec{j} \cdot \vec{j}) + a_z (\vec{j} \cdot \vec{k})$$

$$= a_x (0) + a_y(1) + a_z(0)$$

$$= a_y .$$

Now, if we choose $|\vec{A}| = 1$, we get

$$a_y = \vec{j} \cdot \vec{A} = |\vec{j}| \times |\vec{A}| \cos 90^\circ$$

$$= 0 .$$

Proceeding similarly, we have

$$\vec{i} \cdot \vec{A} = a_x = \cos 150^\circ = - \cos 30^\circ$$

$$= -\frac{\sqrt{3}}{2} .$$

$$\vec{k} \cdot \vec{A} = a_z = \cos 60^\circ = \frac{1}{2} .$$

Hence,

$$\vec{A} = -\frac{\sqrt{3}}{2}\vec{i} + \frac{1}{2}\vec{k} .$$

We now seek a vector $\vec{B}$ which is parallel to cable (2). Let us first find a vector $\vec{u}$ of length one, which lies in the xy-plane directly under cable (2) (i.e., $\vec{u}$ points along the noon-day shadow of cable (2)).

Evidently,

$$\vec{u} = \frac{1}{2}\vec{i} + \frac{\sqrt{3}}{2}\vec{j} .$$

Now, $\vec{B}$ lies in the plane of $\vec{k}$ and $\vec{u}$. Hence, we can use $\vec{k}$ and $\vec{u}$ as basis vectors. Thus,

$$\vec{B} = b_1\vec{k} + b_2\vec{u} .$$

To find $b_1$ and $b_2$, we proceed as before.

Let $|\vec{B}| = 1$.

$$\vec{k} \cdot \vec{B} = b_1 = \cos 60^\circ = \frac{1}{2} .$$

$$\vec{u} \cdot \vec{B} = b_2 = \cos 30^\circ = \frac{\sqrt{3}}{2} .$$

Consequently,

$$\vec{B} = \frac{1}{2}\vec{k} + \frac{\sqrt{3}}{2}\vec{u}$$

$$= \frac{1}{2}\vec{k} + \frac{\sqrt{3}}{2}\left(\frac{1}{2}\vec{i} + \frac{\sqrt{3}}{2}\vec{j}\right)$$

$$= \frac{\sqrt{3}}{4}\vec{i} + \frac{3}{4}\vec{j} + \frac{1}{2}\vec{k} .$$

By symmetry, we can see that $\vec{C}$, the unit vector parallel to cable (3), must be

$$\vec{C} = \frac{\sqrt{3}}{4}\vec{i} - \frac{3}{4}\vec{j} + \frac{1}{2}\vec{k} .$$

The forces are equal in each cable. Let one unit of length of vector represent one pound of force. Then, since the cables are flexible and can transmit only forces parallel to themselves, we have

$$\vec{F}_1 = c\vec{A} ,$$

$$\vec{F}_2 = c\vec{B} ,$$

$$\vec{F}_3 = c\vec{C} .$$

We can now find the scalar $c$. The total upward component is

$$c\left(\frac{1}{2}\vec{k} + \frac{1}{2}\vec{k} + \frac{1}{2}\vec{k}\right) = c\,\frac{3}{2}\vec{k} .$$

But the total upward component must balance the downward force of gravity. Consequently,

$$\frac{3}{2}c = 15 ,$$

$$c = 10 .$$

Thus, $\vec{F}_1 = 10\vec{A}$.

Since $|\vec{A}| = 1$, it follows that $|\vec{F}_1| = 10$. Therefore, there is a tension of 10 pounds in each cable.

Second Solution: Begin exactly as you did in the first solution, but notice that once we have

$$\vec{A} = -\frac{\sqrt{3}}{2}\vec{i} + \frac{1}{2}\vec{k} ,$$

we know that

$$\vec{B} = b_x\vec{i} + b_y\vec{j} + \frac{1}{2}\vec{k} ,$$

$$\vec{C} = c_x\vec{i} + c_y\vec{j} + \frac{1}{2}\vec{k} .$$

We do not need to find x and y components. It suffices to work only with vertical components!

As before,

$$F_1 = cA ,$$

$$F_2 = cB ,$$

$$F_3 = cC ;$$

and we get

$$\frac{3}{2}c = 15 ,$$

$$c = 10 ,$$

$$|\vec{F}_1| = 10 .$$

Hence, we again find that each cord exerts a force of 10 pounds on the lighting fixture.

6. Let us choose axes so that the xy-plane is horizontal, with $\vec{j}$ pointing north and $\vec{i}$ pointing east. The three vectors we need to consider are as follows:

$\vec{A}_G$ (representing the velocity of the airplane with respect to the ground) ;

$\vec{A}_W$ (representing the velocity of the airplane with respect to the air) ; and

$\vec{W}_G$ (representing the velocity of the wind with respect to the ground) .

We know from physics that

$$\vec{A_G} = \vec{A_W} + \vec{W_G} .$$

Now, $\vec{A_W} = 100\,[(\cos 30^\circ)\vec{j} + (\cos 60^\circ)\vec{k}]$

$= 50\sqrt{3}\,\vec{j} + 50\,\vec{k}$ ;

also, $\vec{W_G} = 30\,\vec{i}$ .

Consequently,

$$\vec{A_G} = 30\,\vec{i} + 50\sqrt{3}\,\vec{j} + 50\,\vec{k} .$$

The upward component $50\,\vec{k}$ does not appear in the ground speed. In fact, the ground speed is

$$|\vec{A_G} - 50\,\vec{k}| = \sqrt{30^2 + (50\sqrt{3})^2}$$

$$= \sqrt{8400}$$

$$= 92 \text{ miles per hour.}$$

7. Evidently, the pilot will achieve the <u>fastest</u> ground speed if his heading is <u>with the wind</u>. Using the notation employed in Problem 6, we have

$$\vec{A_W} = 50\sqrt{3}\,\vec{i} + 50\,\vec{k} ,$$

$$\vec{A_G} = (30 + 50\sqrt{3})\,\vec{i} + 50\,\vec{k} ,$$

$$|\vec{A_G} - 50\,\vec{k}| = \sqrt{(30 + 50\sqrt{3})^2}$$

$$= 117 \text{ m.p.h.}$$

Similarly, the <u>smallest</u> ground speed will be achieved if the pilot heads into the wind; in this case the ground speed will be

$$|\vec{A_G} - 50\,\vec{k}| = \sqrt{(50\sqrt{3} - 30)^2}$$

$$= 57 \text{ m.p.h.}$$

8. The proof is analogous to the one for two dimensions.
9. $7\,\vec{i} - 3\,\vec{j} + 5\,\vec{k}$ .
10. $\frac{1}{\sqrt{170}}$ .
11. $\frac{1}{\sqrt{14}}$ .

## 11-6. Vectors as a Formal Mathematical System.

The main topic of this section is the solution of a problem. To teach this section successfully the teacher must do more than solve the problem. He must help the student understand what the problem is and also help him understand that which is offered as a solution of the problem really solves the problem.

First, let us consider what the problem is. We learned that vectors obey certain rules. We ask whether vectors are the only objects which obey these rules. The answer is certainly "no", since forces and velocities also obey them. The question which we propose is whether any system of objects which obeys these rules can be correctly treated as a system of vectors--whether it is "essentially the same" as our system of vectors. We answer this question by proving that any system which obeys Rules 1-11 is isomorphic to our system of vectors. The question as to whether systems which are "isomorphic" are really "essentially the same" systems is a philosophical one. It should not be completely by-passed but it cannot be answered beyond a shadow of doubt.

### Exercises 11-6. Answers.

1. Yes. In this case Rules 1-11 are restatements of Rules $C_1$-$C_2$ of Chapter V, Section 1.
2. The system obeys the rules 1, 2, 3, 4, 5, 6, 8, 9, 11, but not 7 and not 10.

   The left member of Rule 7 becomes

$$r \odot (s \odot (a,b)) = r \odot (\tfrac{sa}{2}, \tfrac{sb}{2})$$
$$= (\frac{r(\frac{sa}{2})}{2}, \frac{r(\frac{sb}{2})}{2})$$
$$= (\tfrac{rsa}{4}, \tfrac{rsb}{4})$$

   and the right member of Rule 7 becomes

$$(rs) \odot (a,b) = (\tfrac{rs}{2}a, \tfrac{rs}{2}b).$$

   These are not equal.

The left member of Rule 10 becomes

$$1 \odot (a,b) = (\tfrac{a}{2} , \tfrac{b}{2})$$

The right member of Rule 10 becomes $(a,b)$ . These are not equal.

3. This system obeys rules 1,2,5,6,7,9,10,11, but not 3 , not 4 , and not 8 .

The left member of Rule 3 becomes

$$(a,b) \oplus ((c,d) \oplus (e,f)) = (a,b) \oplus (\tfrac{c + e}{2} , \tfrac{d + f}{2})$$

$$= (\tfrac{2a + c + e}{4} , \tfrac{2b + d + f}{4}) .$$

The right member of Rule 3 becomes

$$((a,b) \oplus (c,d)) + (e,f) = (\tfrac{a + c}{2} , \tfrac{b + d}{2}) \oplus (e,f)$$

$$= (\tfrac{a + c + 2e}{2} , \tfrac{b + d + 2f}{2}) .$$

These are not equal.

The left member of Rule 4 becomes

$$(a,b) \oplus (x,y) = (\tfrac{a + x}{2} , \tfrac{b + y}{2}) .$$

The right member of Rule 4 is $(a,b)$ .

These two are equal if and only if $x = a$ and $y = b$ .

Therefore there is no single $(x,y)$ such that for all $(a,b)$ ,

$$(a,b) \oplus (x,y) = (a,b) .$$

The left member of Rule 8 becomes

$$(r + s) \odot (a,b) = ((r + s)a , (r + s)b) .$$

The right member of Rule 8 becomes

$$(r \odot (a,b)) \oplus (s \odot (a,b)) = (ra,rb) \oplus (sa,sb)$$

$$= (\tfrac{ra + sa}{2} , \tfrac{rb + sb}{2}) .$$

These are not equal.

## 11-7. Illustrative Test Questions.

1. If ABCD is a parallelogram which of the following are pairs of parallel rays?
   (a) $\overrightarrow{AB}$ , $\overrightarrow{CD}$ .
   (b) $\overrightarrow{AD}$ , $\overrightarrow{BC}$ .
   (c) $\overrightarrow{AC}$ , $\overrightarrow{BD}$ .
   (d) $\overrightarrow{AB}$ , $\overrightarrow{AB}$ .
2. Which of the following are true statements?
   (a) If $\overrightarrow{AB}$ , $\overrightarrow{CD}$ are parallel rays and $\overrightarrow{AB}$ , $\overrightarrow{EF}$ are parallel rays, then $\overrightarrow{CD}$ , $\overrightarrow{EF}$ are parallel rays.
   (b) If $\overrightarrow{AB}$ , $\overrightarrow{CD}$ are parallel rays. then $\overrightarrow{BA}$ , $\overrightarrow{DC}$ are parallel rays.
   (c) If $\overrightarrow{AB}$ , $\overrightarrow{CD}$ are parallel rays, then $\overrightarrow{AC}$ , $\overrightarrow{BD}$ are parallel rays.
   (d) If $\overrightarrow{AB}$ , $\overrightarrow{CD}$ are <u>not</u> parallel rays and $\overrightarrow{AB}$ , $\overrightarrow{EF}$ are <u>not</u> parallel rays, then $\overrightarrow{CD}$ , $\overrightarrow{EF}$ are <u>not</u> parallel rays.
3. If ABCD is a parallelogram which of the following are pairs of equivalent directed line segments?
   (a) $\overrightarrow{AB}$ , $\overrightarrow{CD}$ .
   (b) $\overrightarrow{AB}$ , $\overrightarrow{DC}$ .
   (c) $\overrightarrow{BA}$ , $\overrightarrow{CD}$ .
   (d) $\overrightarrow{BA}$ , $\overrightarrow{DC}$ .
   (e) $\overrightarrow{BD}$ , $\overrightarrow{AC}$ .
   (f) $\overrightarrow{BD}$ , $\overrightarrow{CA}$ .
   (g) $\overrightarrow{BD}$ , $\overrightarrow{DB}$ .
4. Which of the following are true statements?
   (a) If $\overrightarrow{AB}$ , $\overrightarrow{CD}$ are equivalent directed line segments and if $\overrightarrow{AB}$ , $\overrightarrow{EF}$ are equivalent directed line segments, then $\overrightarrow{CD}$ , $\overrightarrow{EF}$ are equivalent directed line segments.
   (b) If $\overrightarrow{AB}$ , $\overrightarrow{CD}$ are equivalent directed line segments, then $\overrightarrow{BA}$ , $\overrightarrow{DC}$ are equivalent directed line segments.

(c) If $\overrightarrow{AB}$ , $\overrightarrow{CD}$ are equivalent directed line segments, then $\overrightarrow{AC}$ , $\overrightarrow{BD}$ are equivalent directed line segments.

(d) If $\overrightarrow{AB}$ , $\overrightarrow{CD}$ are not equivalent directed line segments and if $\overrightarrow{AB}$ , $\overrightarrow{EF}$ are not equivalent directed line segments, then $\overrightarrow{CD}$ , $\overrightarrow{EF}$ are not equivalent directed line segments.

5. ABCD is a parallelogram and P , Q , R , S are the midpoints of its sides. (See Figure 11-7a.) Show that each of the directed line segments $\overrightarrow{OB}$ , $\overrightarrow{OC}$ , $\overrightarrow{OD}$ , $\overrightarrow{OA}$ is equivalent to a sum of two of the directed line segments $\overrightarrow{OP}$ , $\overrightarrow{PO}$ , $\overrightarrow{OQ}$ , $\overrightarrow{QO}$ .

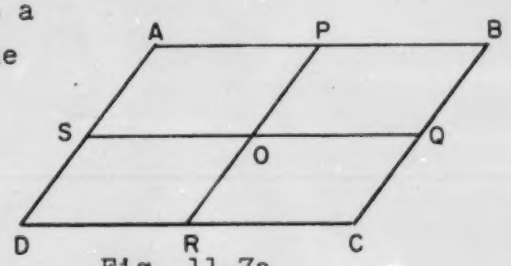
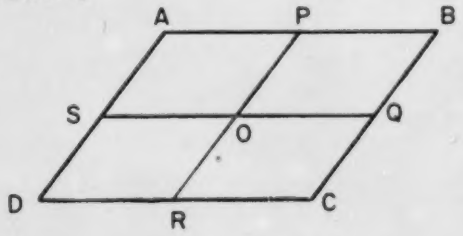


Fig. 11-7a.

6. Points A , B , C , D on the number line have respective coordinates -2 , -1 , 0 , 1 . Find t so that

(a) $\overrightarrow{AB} \doteq t\,\overrightarrow{CD}$ .
(b) $\overrightarrow{AB} \doteq t\,\overrightarrow{DC}$ .
(c) $\overrightarrow{BA} \doteq t\,\overrightarrow{CD}$ .
(d) $\overrightarrow{BA} \doteq t\,\overrightarrow{DC}$ .
(e) $\overrightarrow{AC} \doteq t\,\overrightarrow{AD}$ .
(f) $\overrightarrow{AC} \doteq t\,\overrightarrow{CB}$ .

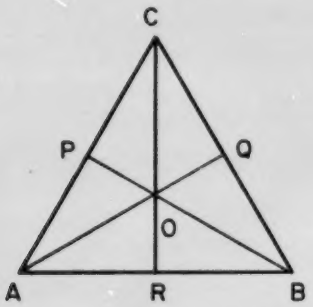
7. ABCD is a parallelogram and P , Q , R , S are the midpoints of its sides.




For each of the following directed line segments, find an equivalent directed line segment of the form $r\,\overrightarrow{OQ} + s\,\overrightarrow{OP}$ .

(a) $\overrightarrow{OB}$ .
(b) $\overrightarrow{DQ}$ .
(c) $\overrightarrow{DC}$ .
(d) $\overrightarrow{AO}$ .
(e) $\overrightarrow{PQ}$ .
(f) $\overrightarrow{RS}$ .
(g) $\overrightarrow{DB}$ .
(h) $\overrightarrow{CA}$ .

8. P , Q , R are the midpoints of the sides of triangle ABC .




For each of the following directed line segments, find an equivalent directed line segment of the form $r\overrightarrow{AB} + s\overrightarrow{AC}$ .

(a) $\overrightarrow{BC}$ .
(b) $\overrightarrow{CB}$ .
(c) $\overrightarrow{BO}$ .
(d) $\overrightarrow{PC}$ .
(e) $\overrightarrow{BR}$ .
(f) $\overrightarrow{OQ}$ .
(g) $\overrightarrow{OC}$ .
(h) $\overrightarrow{OP}$ .

9. If A , B , C are respectively (2,1) , (3,4) , (-1,2) find X so that

(a) $\overrightarrow{AB} \doteq \overrightarrow{CX}$ .
(b) $\overrightarrow{AX} \doteq \overrightarrow{CB}$ .
(c) $\overrightarrow{XA} \doteq \overrightarrow{CB}$ .
(d) $\overrightarrow{XA} \doteq \overrightarrow{BC}$ .

10. Find the components of

(a) [4,-1] + [-5,2] .
(b) 6[4,-1] + 6[-5,2] .
(c) 6([4,-1] + [-5,2]) .

11. Find the components of

(a) -[5,-6] .
(b) -1[5,-6] .
(c) 5[5,-6] .
(d) 0[5,-6] .
(e) 5[5,-6] + 2[5,-6] .

12. Determine x and y so that

$$x[4,2] + y[5,1] = [3,3] .$$

13. Determine $x$ and $y$ so that

$$x[4,2] + y[2,1] = [0,0] .$$

(Infinitely many solutions.)

14. Determine $a$ and $b$ so that

(a) $[4,3] = a\vec{i} + b\vec{j}$ .  (c) $\vec{i} = a[4,3] + b[3,4]$ .

(b) $[3,4] = a\vec{i} + b\vec{j}$ .  (d) $\vec{j} = a[4,3] + b[3,4]$ .

15. Determine $a$ and $b$ so that

$$5\vec{i} + 6\vec{j} = a(2\vec{i} + 3\vec{j}) + b(3\vec{i} - 2\vec{j}) .$$

16. Find $\vec{X} \cdot \vec{Y}$ if

(a) $\vec{X} = 2\vec{i} + 3\vec{j}$ , $\vec{Y} = -4\vec{i} + 5\vec{j}$ .

(b) $\vec{X} = \vec{i}$ , $\vec{Y} = -4\vec{i} + 5\vec{j}$ .

(c) $\vec{X} = \vec{j}$ , $\vec{Y} = -4\vec{i} + 5\vec{j}$ .

(d) $\vec{X} = 2\vec{i}$ , $\vec{Y} = 5\vec{j}$ .

17. Given that $\vec{X} \cdot \vec{i} = 2$ and $\vec{X} \cdot \vec{j} = 3$ . Find $a$ and $b$ so that

$$\vec{X} = a\vec{i} + b\vec{j} .$$

18. Find the angle between $\vec{X}$ and $\vec{Y}$ if $|\vec{X}| = 3$ , $|\vec{Y}| = 4$ and $\vec{X} \cdot \vec{Y}$ is

(a) 0 .  (c) 4 .

(b) -3 .  (d) -12 .

19. Find the component of $\vec{Y}$ in the direction of $\vec{X}$ if

$$\vec{X} = 2\vec{i} + 3\vec{j} , \vec{Y} = 4\vec{i} - 2\vec{j} .$$

20. Find $\vec{X} \cdot \vec{Y}$ if

(a) $\vec{X} = 2\vec{i} + 3\vec{j} + 4\vec{k}$ , $\vec{Y} = -2\vec{i} - 2\vec{j} - 2\vec{k}$ .

(b) $\vec{X} = \vec{i}$ , $\vec{Y} = 2\vec{i} + 3\vec{j} + 4\vec{k}$ .

(c) $\vec{X} = \vec{i} + \vec{j}$ , $\vec{Y} = 2\vec{i} + 3\vec{j} + 4\vec{k}$ .

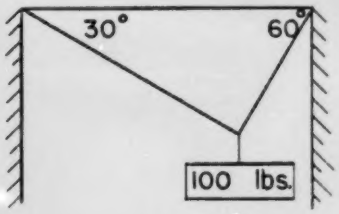
(d) $\vec{X} = \vec{i} + \vec{j}$ , $\vec{Y} = 3\vec{j} + 4\vec{k}$ .

21. Find the angle between $\vec{X}$ and $\vec{Y}$ if

$$\vec{X} = \vec{i} + \vec{j} + \vec{k}\ ,\ \vec{Y} = 2\vec{i} + 3\vec{j} + 4\vec{k}\ .$$

22. If a vector $\vec{V}$ of unit length makes an angle of $60^\circ$ with the x-axis, write an expression for $\vec{V}$ in the form $a\vec{i} + b\vec{j}$ .

23. The accompanying figure shows a weight of 100 pounds suspended by two cords that make angles of $30^\circ$ and $60^\circ$ with the horizontal. Find the tension in each cord.



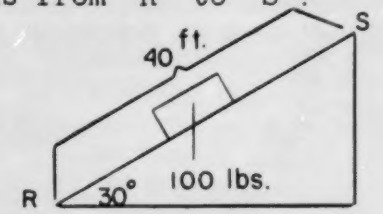

24. An aviator who heads his plane due north at a velocity of 120 miles per hour encounters a hurricane blowing due east at a velocity of 90 miles per hour. Draw a diagram showing his path of flight and position at the end of one hour.

25. Which of the following is correct?
Three forces which are represented by the vectors A, B, and C are in equilibrium if and only if

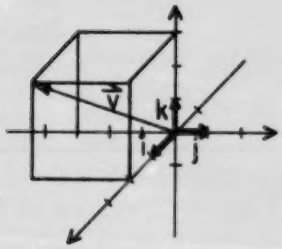
(a) $\vec{A} + \vec{B} = \vec{C}$ .
(b) $\vec{A} = -\vec{B}$ .
(c) $\vec{C} = -(\vec{A} + \vec{B})$ .
(d) $\vec{A}$ is not coplanar with $\vec{B}$ and $\vec{C}$ .
(e) $\vec{A}$ is perpendicular to $\vec{B}$ and $\vec{C}$ .

26. Which of the following is correct?
The drawing shows a smooth incline 40 feet long that makes an angle of $30^\circ$ with the horizontal. How much work is done in moving an object of 100 pounds from R to S ?

(a) 2000 foot pounds.
(b) $2000\sqrt{3}$ foot pounds.
(c) $4000/\sqrt{3}$ foot pounds.
(d) 4000 foot pounds.
(e) None of the above is correct.



27. Which of the following is correct?
In the figure shown in the drawing $|\vec{V}|$ is equal to




(a) 2 .
(b) $2\sqrt{2}$ .
(c) $\sqrt{22}$ .
(d) 22 .
(e) $2\vec{i} - 3\vec{j} + 3\vec{k}$ .

28. If $(a,b) \oplus (c,d)$ is defined to be $(2a + 3c , 2b + 3d)$ for all pairs $(a,b)$ , $(c,d)$ of real numbers, which of the following is correct?

(a) $(a,b) \oplus (c,d) = (c,d) \oplus (a,b)$ .
(b) $(a,b) \oplus ((c,d) \oplus (e,f)) = ((a,b) \oplus (c,d)) \oplus (e,f)$ .
(c) There is a single pair $(a,b)$ such that for all pairs $(a,b)$

$$(a,b) \oplus (u,v) = (a,b) .$$

(d) For each pair $(a,b)$ there is a pair $(u,v)$ such that

$$(a,b) \oplus (u,v) = (0,0) .$$

11-7. Illustrative Test Questions. Answers.

1. b and d .

2. (a) T .
(b) T .
(c) F .
(d) F .

3. b and c .

4. (a) T .
(b) T .
(c) T .
(d) F .

5. $\vec{OB} \doteq \vec{OQ} + \vec{OP}$
$\vec{OC} \doteq \vec{OQ} + \vec{PO}$
$\vec{OD} \doteq \vec{QO} + \vec{PO}$
$\vec{OA} \doteq \vec{QO} + \vec{OP}$

6. (a) 1 .
(b) -1 .
(c) -1 .
(d) 1 .
(e) $\frac{2}{3}$ .
(f) -2 .

7. (a) $\vec{OQ} + \vec{OP}$ .
(b) $\vec{OQ} + \vec{OP}$ .
(c) $2\vec{OQ} + \vec{OP}$ .
(d) $\vec{OQ} - \vec{OP}$ .
(e) $\vec{OQ} - \vec{OP}$ .
(f) $-\vec{OQ} + \vec{OP}$ .
(g) $2\vec{OQ} + 2\vec{OP}$ .
(h) $-2\vec{OQ} + 2\vec{OP}$ .

8. (a) $-\overrightarrow{AB} + \overrightarrow{AC}$ .
   (b) $\overrightarrow{AB} - \overrightarrow{AC}$ .
   (c) $-\frac{2}{3}\overrightarrow{AB} + \frac{1}{3}\overrightarrow{AC}$ .
   (d) $0 \cdot \overrightarrow{AB} + \frac{1}{2}\overrightarrow{AC}$ .
   (e) $-\frac{1}{2}\overrightarrow{AB} + 0 \cdot \overrightarrow{AC}$ .
   (f) $\frac{1}{6}(\overrightarrow{AB} + \overrightarrow{AC})$ .
   (g) $-\frac{1}{3}\overrightarrow{AB} + \frac{2}{3}\overrightarrow{AC}$ .
   (h) $-\frac{1}{3}\overrightarrow{AB} + \frac{1}{6}\overrightarrow{AC}$ .

9. (a) (0,5) .
   (b) (6,3) .
   (c) (-2,-1) .
   (d) (6,3) .

10. (a) [-1,1] .
    (b) [-6,6] .
    (c) [-6,6] .

11. (a) [-5,6] .
    (b) [-5,6] .
    (c) [25,-30] .
    (d) [0,0] .
    (e) [35,-42] .

12. $x = 2$ , $y = -1$ .

13. $y$ is arbitrary , $x = -\frac{1}{2}y$ .

14. (a) $a = 4$ , $b = 3$ .
    (b) $a = 3$ , $b = 4$ .
    (c) $a = \frac{4}{7}$ , $b = -\frac{3}{7}$ .
    (d) $a = -\frac{3}{7}$ , $b = \frac{4}{7}$ .

15. $a = \frac{28}{13}$ , $b = \frac{3}{13}$ .

16. (a) 7 . (c) 5 .
    (b) -4 . (d) 0 .

17. a = 2 , b = 3 .

18. (a) $90^\circ$ . (c) $70.5^\circ$ .
    (b) $104.5^\circ$ . (d) $180^\circ$ .

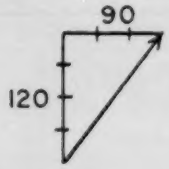
19. $\frac{2}{\sqrt{13}}$ .

20. (a) -18 . (c) 5 .
    (b) 2 . (d) 3 .

21. $15.2^\circ$ .

22. $\vec{i} + \sqrt{3}\,\vec{j}$ .

23. $|\vec{R}| = 50$ , $|\vec{S}| = 50\sqrt{3} \approx 86.6$ .

24. 


25. c .

26. b .

27. e .

28. c,d .

# Chapter 12

## POLAR FORM OF COMPLEX NUMBERS

### 12-1. Introduction.

In this chapter we introduce and study the "polar" representation of complex numbers.

The central theorem for all this work is the theorem of de Moivre. We indicate in the text an "induction" proof. In this commentary we supply the details. The relation between mathematical induction and the well order property of the natural numbers is discussed in Section 1-3 of this Commentary.

The proof. We wish to show that, for every natural number $n$,

$$(*) \qquad (\cos\theta + i \sin\theta)^n = \cos n\theta + i \sin n\theta .$$

Let us suppose--contrary to our desired conclusion--that formula (*) is false for one or more natural numbers $n$. We intend to force a contradiction from this supposition. Our supposition states that the set of natural numbers $n$ for which (*) is false contains one or more members. The well order property asserts that such a set contains a minimal member $m$; an element $m$ such that every natural number less than $m$ is <u>not</u> in the set and each member of the set is greater than or equal to $m$. (Note we cannot say conversely that every natural number greater than $m$ is in the set.)

Now there are two possibilities: either $m = 1$ or $m > 1$. But $m$ cannot be $1$ for, with $m = 1$, (*) says

$$\cos\theta + i \sin\theta = \cos\theta + i \sin\theta$$

and this is certainly true (by Property $\underline{E}_2$ of Chapter 1).

Hence $m > 1$ . But if $m > 1$ , then (i) $m - 1$ is a natural number, and (ii) (*) is true for $n = m - 1$ since $m - 1 < m$ . We know, then, that

(**) $(\cos\theta + i \sin\theta)^{m-1} = \cos(m - 1)\theta + i \sin(m - 1)\theta$

is TRUE. Multiplying both sides of (**) by $\cos\theta + i \sin\theta$ we get

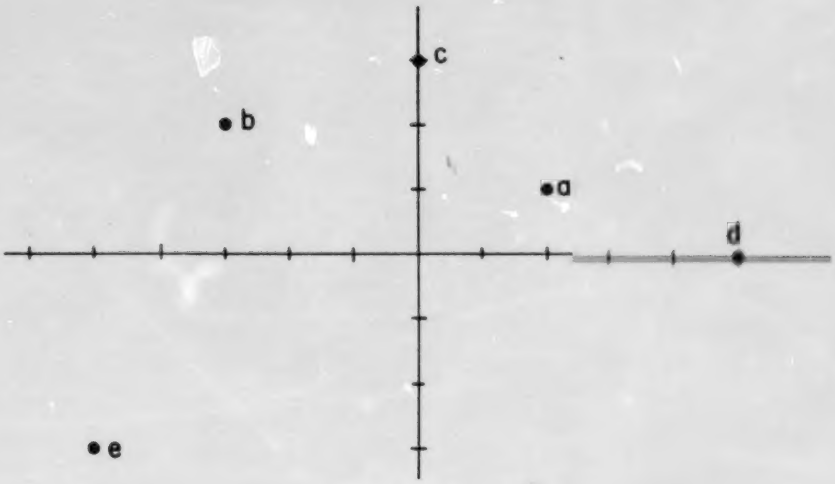
$$\begin{aligned}(\cos\theta + i \sin\theta)^{m} &= [\cos(m - 1)\theta + i \sin(m - 1)\theta](\cos\theta + i \sin\theta) \\ &= [\cos(m - 1)\theta \cos\theta - \sin(m - 1)\theta \sin\theta] \\ &\quad + i[\sin(m - 1)\theta \cos\theta + \cos(m - 1)\theta \sin\theta] \\ &= \cos m\theta + i \sin m\theta \quad .\end{aligned}$$

which must also be true. But $m$ is one of the numbers for which (*) is false. This is the contradiction. We are forced to discard our supposition and conclude that (*) is true for each natural number $n$ .

Exercises 12-1. Answers.

1. (a) $1 + 0 \cdot i$ . (d) $0 + 0 \cdot i$ .
   (b) $0 + 1 \cdot i$ . (e) $0 + 4i$ .
   (c) $(\frac{2}{13}) + (-\frac{3}{13})i$ . (f) $0 + \sqrt{7}\, i$ .
2. (a) $2 + 3i$ .
   (b) $5 - i$ .
   (c) $-2 - 3i$ .
3. $(a^2 + b^2) + 0 \cdot i$ .
4. $0 + (-1)i$ .
5. (a) $5$ .
   (b) $\sqrt{29}$ .
   (c) $3$ .
6. $z = (-\frac{1}{4}) + (\frac{1}{4}\sqrt{7})i$ or $z = (-\frac{1}{4}) - (\frac{1}{4}\sqrt{7})i$ .

7.




8. (a) $-2 - 2i$ .
   (b) $-2 + i$ .
   (c) $-2 + 5i$ .
   (d) $1 + 2i$ .
   (e) $7 - i$ .
   (f) $6 - 9i$ .
   (g) $5 + i$ .
   (h) $5$ .

9. $a = b$ .

10. (a) $x = y = 2$ .
    (b) $x = 2$ , $y = -1$ .
    (c) $x = 7$ , $y = 11$ .
    (d) $x = 2$ . ($-2$ extraneous)

11. (a) $(a + bi)(x + yi) = (ax - by) + (ay + bx)i = 1$

$$ax - by = 1$$
$$bx + ay = 0$$

$$x = \frac{a}{a^2 + b^2} \; ; \; y = -\frac{b}{a^2 + b^2} \; .$$

(b) $\frac{1}{a + bi} \cdot \frac{a - bi}{a - bi} = \left(\frac{a}{a^2 + b^2}\right) - \left(\frac{b}{a^2 + b^2}\right)i$ .

12. $\frac{1}{1 + z^2}$ .

Exercises 12-2. Answers.

1. (a) $2\sqrt{2}\left(\frac{1}{\sqrt{2}} + 1\frac{1}{\sqrt{2}}\right) = 2\sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$ ; $\theta = \frac{\pi}{4}$ .

(b) $3\sqrt{2}\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = 3\sqrt{2}(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})$ ; $\theta = \frac{3\pi}{4}$ .

(c) $\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right) = \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}$ ; $\theta = \frac{5\pi}{3}$ .

(d) $-\frac{\sqrt{3}}{2} + i\left(\frac{1}{2}\right) = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}$ ; $\theta = \frac{5\pi}{6}$ .

(e) $4(1 + 0i) = 4(\cos 0 + i\sin 0)$ ; $\theta = 0$ .

(f) $2(0 - i) = 2(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2})$ ; $\theta = \frac{3\pi}{2}$ .

2. (a) $\frac{3}{2} + \frac{3\sqrt{3}}{2}i$ .

(b) $\sqrt{2}(-1 + i)$ .

(c) $\frac{\sqrt{3}}{2} + (-\frac{1}{2})i$ .

(d) $-5 + 0i$ .

(e) $0 + i$ .

(f) $2 + 0i$ .

3. (a) $6(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}) = -3\sqrt{3} + 3i$ .

(b) $\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$ .

(c) $9(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}) = 0 + 9i$ .

4. (See Section 12-3.)

5. $$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \cdot \frac{\cos\theta_2 - i\sin\theta_2}{\cos\theta_2 - i\sin\theta_2}$$

$$= \frac{r_1}{r_2}[(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)]$$

$$= \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)] .$$

6. Using Formula 5-8c: $z\bar{z} = |z|^2$ , $|z| = 1$ gives $z\bar{z} = 1$.

Hence $\bar{z} = \frac{1}{z}$ . (Note that $z \neq 0$ since $|z| \neq 0$.)

Using polar form: $|z| = 1$ gives $z = \cos\theta + i \sin\theta$ .

Then $$\frac{1}{z} = \frac{1}{\cos\theta + i \sin\theta} = \frac{\cos\theta - i \sin\theta}{(\cos\theta)^2 + (\sin\theta)^2}$$

$$= \cos\theta - i \sin\theta = \bar{z} .$$

7. According to exercise five

(1) $$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)]$$

$z_1$ and $z_2$ lie on the same ray from 0 if and only if $\theta_1 - \theta_2 = 2\pi k$ where $k$ is an integer.

If $\theta_1 - \theta_2 = 2\pi k$ then $\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos 2\pi k + i \sin 2\pi k]$

$= \frac{r_1}{r_2}(1 + 0) = \frac{r_1}{r_2}$ a real number.

$\therefore$ If $z_1$ and $z_2$ lie on the same ray from 0, then their quotient is a real number. Conversely if the quotient of $z_1$ and $z_2$ is a real number then the coefficient of $i$ in the right member of (1) must be zero.

$\frac{r_1}{r_2} \sin(\theta_1 - \theta_2) = 0$. Now $r_1 \neq 0$ by hypothesis.

Hence $\sin(\theta_1 - \theta_2) = 0$ . This is true if and only if $\theta_1 - \theta_2 = 2\pi k$ which is precisely the condition cited above for 0, $z_1$ and $z_2$ to be collinear.

8. $\cos\phi + i \sin\phi = \cos\theta + i \sin\theta$ if and only if

$\frac{\cos\phi + i \sin\phi}{\cos\theta + i \sin\theta} = \cos(\phi - \theta) + i \sin(\phi - \theta) = 1 + 0i$

if and only if (c) $\cos(\phi - \theta) = 1$ and $\sin(\phi - \theta) = 0$ .

Now $\cos(\phi - \theta) = 1$ implies

$$[\sin(\phi - \theta)]^2 = 1 - [\cos(\phi - \theta)]^2 = 1 - 1 = 0 ,$$

but $\sin(\phi - \theta) = 0$ gives only

$$[\cos(\phi - \theta)]^2 = 1 - [\sin(\phi - \theta)]^2 = 1 .$$

Thus $\cos(\phi - \theta) = 1$ implies $\sin(\phi - \theta) = 0$ ; but $\sin(\phi - \theta) = 0$ implies only $\cos(\phi - \theta) = 1$ or $\cos(\phi - \theta) = -1$ . In any case, the pair of conditions (c) above is equivalent to the single condition $\cos(\phi - \theta) = 1$ . But $\cos(\phi - \theta) = 1$ if and only if

$\phi - \theta = 2k\pi$ , where $k$ is an integer.

Exercises 12-3. Answers.

1. $|z| = \sqrt{2}$ , $\arg z = \frac{\pi}{4}$ ; $z = \sqrt{2}(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4})$ .

$z^2 = 2(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}) = 2i$ .

$z^3 = 2\sqrt{2}(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}) = -2 + 2i$ .

$z^4 = 4(\cos \pi + i \sin \pi) = -4$ .

2. $|z| = 1$ , $\arg z = \frac{2\pi}{3}$ ; $z = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$ .

$z^2 = \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$ .

$z^3 = \cos 2\pi + i \sin 2\pi = 1$ .

$z^4 = \cos \frac{8\pi}{3} + i \sin \frac{8\pi}{3}$ ,

$= \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$ ,

$= z = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ .

3. $|z| = \frac{1}{3}$ , $\arg z = \frac{\pi}{2}$ ; $z = \frac{1}{3}(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2})$ .

$z^2 = \frac{1}{9}(\cos \pi + i \sin \pi) = -\frac{1}{9}$ .

$z^3 = \frac{1}{27}(\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2}) = -\frac{1}{27}$ .

$z^4 = \frac{1}{81}(\cos 2\pi + i \sin 2\pi) = \frac{1}{81}$ .

4. $|z| = 5$ , $\arg z = \theta$ , where $\cos \theta = \frac{3}{5}$ and $\sin \theta = \frac{4}{5}$ .
($\theta \approx 52^\circ$ .)

$z = 5(\cos \theta + i \sin \theta)$ .

$z^2 = 25(\cos 2\theta + i \sin 2\theta) = 7 + 24i$

$\approx 25(\cos 104^\circ + i \sin 104^\circ)$ .

$z^3 = 125(\cos 3\theta + i \sin 3\theta) = -117 + 44i$

$\approx 125(\cos 156^\circ + i \sin 156^\circ)$ .

$z^4 = 625(\cos 4\theta + i \sin 4\theta) = -527 - 336i$

$\approx 625(\cos 208^\circ + i \sin 208^\circ)$ .

5. $|z| = 1$ , $\arg z = \frac{4\pi}{3}$ .

$z = \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$ .

$z^2 = \cos \frac{8\pi}{3} + i \sin \frac{8\pi}{3}$

$= \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ .

$z^3 = \cos 4\pi + i \sin 4\pi$

$= \cos 0 + i \sin 0 = 1$ .

$z^4 = \cos \frac{16\pi}{3} + i \sin \frac{16\pi}{3}$

$= \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3} = z = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$ .

6. $z^{-n} = \frac{1}{z^n} = \frac{1}{r^n(\cos n\theta + i \sin n\theta)}$

$= \frac{1}{r^n}(\cos n\theta - i \sin n\theta)$

$= r^{-n}[\cos(-n)\theta + i \sin(-n)\theta]$ .

Exercises 12-4. Answers.

1. $w_0 = 2$ , $w_1 = -2$ .

2. $w_0 = \sqrt{\sqrt{2} + 1} + i\sqrt{\sqrt{2} - 1}$

$w_1 = -\sqrt{\sqrt{2} + 1} - i\sqrt{\sqrt{2} - 1}$ .

3. $w_0 = 3i$ , $w_1 = -3i$ .

4. $w_0 = \frac{\sqrt{2}}{2}[-\sqrt{2 + \sqrt{3}} + i\sqrt{2 - \sqrt{3}}]$

$w_1 = -w_0$ .

5. $w_0 = 2 + i$ , $w_1 = -w_0$ .

6. $w_0 = \sqrt{\frac{\sqrt{5} + \sqrt{2}}{2}} + i\sqrt{\frac{\sqrt{5} - \sqrt{2}}{2}}$ .

7. $w_{2k} = \sqrt{|z|}\,[\cos(\frac{\theta}{2} + 2k\pi) + i \sin(\frac{\theta}{2} + 2k\pi)]$

$= \sqrt{|z|}\,[\cos(\frac{\theta}{2}) + i \sin(\frac{\theta}{2})] = w_0$

$w_{2k+1} = \sqrt{|z|}\,[\cos[\frac{\theta}{2} + (2k + 1)\pi] + i \sin[\frac{\theta}{2} + (2k + 1)\pi]]$

$= \sqrt{|z|}\,[\cos(\frac{\theta}{2} + \pi) + i \sin(\frac{\theta}{2} + \pi)] = w_1$ .

Exercises 12-5. Answers.

1. $2i$ , $-1$ .

2. $-i$ , $-1$ .

3. $1$ .

4. $1 + i$ .

5. $0\ ,\ -1 + \frac{1}{\sqrt{2}}\left(-\sqrt{\sqrt{10}-1} + i\sqrt{\sqrt{10}+1}\right)\ ,$

$\quad -1 - \frac{1}{\sqrt{2}}\left(-\sqrt{\sqrt{10}-1} + i\sqrt{\sqrt{10}+1}\right)\ .$

6. $\sqrt[4]{2}(-1+i)\ ,\ \sqrt[4]{2}(1-i)$

$\quad B^2 - 4AC = 0\ .$

*7. $w = \pm\frac{1}{2}\sqrt{13}\left(-\frac{2}{\sqrt{13}} + \frac{3i}{\sqrt{13}}\right) = \pm\left(-1 + \frac{3}{2}i\right)$

$z^2 = -1 + 2i \quad \text{or} \quad 1 - i$

$z_0 = 0.7861 + 1.2721i$

$z_1 = -z_0$

$z_2 = -1.099 + 0.4551i$

$z_3 = -z_2\ .$

8. $(z^3 - iz^2) - (1+2i)(z^2 - iz) - (iz + 1)$

$= z^2(z-i) - (1+2i)z\,(z-i) - i(z-1)$

$= (z-i)[z^2 - (1+2i)z - i] = 0$

$z = i\ ,\ \frac{1 + (2+\sqrt{3})i}{2}\ ,\ \frac{1 + (2-\sqrt{3})i}{2}\ .$

Exercises 12-6. Answers.

1. $\sqrt[3]{2}\ ,\ \omega\sqrt[3]{2} = \frac{\sqrt[3]{2}}{2}(-1 + i\sqrt{3})\ ,\ \omega^2\sqrt[3]{2} = \frac{\sqrt[3]{2}}{2}(-1 - i\sqrt{3})\ .$

2. $\frac{\sqrt[3]{2}}{2}(1 + i\sqrt{3})\ ,\ -\sqrt[3]{2}\ ,\ \frac{\sqrt[3]{2}}{2}(+1 - i\sqrt{3})\ .$

3. $\frac{\sqrt{3}+i}{2}\ ,\ \frac{i-\sqrt{3}}{2}\ ,\ -i\ .$

4. $i\ ,\ -\frac{\sqrt{3}+i}{2}\ ,\ \frac{\sqrt{3}-i}{2}\ .$

5. $\sqrt[6]{2}(1+i)\ ,\ \omega\sqrt[6]{2}(1+i)\ ,\ \omega^2\sqrt[6]{2}(1+i)\ .$

6. $w_0 \approx 1.631 + 0.5204\ i$

$w_1 \approx -1.593 + 1.153\ i$

$w_2 \approx -3.637 - 1.671\ i$ .

*7. $\arg(1 + i) = \frac{\pi}{4} = 45^\circ$ .

$\frac{1}{3}\arg(1 + i) = \frac{\pi}{12} = 15^\circ = \frac{1}{2}\arg\left(\frac{\sqrt{3} + i}{2}\right)$ .

$|1 + i| = \sqrt{2}$ . Therefore $|w_0| = \sqrt[6]{2}$ and

$w_0 = \frac{\sqrt[6]{2}}{2}(\sqrt{2 + \sqrt{3}} + i\sqrt{2 - \sqrt{3}})$ ; $w_1 = \frac{1}{2^{\frac{1}{3}}}(-1 + i)$ ;

$w_2 = \frac{\sqrt[6]{2}}{2}(-\sqrt{2 - \sqrt{3}} - i\sqrt{2 + \sqrt{3}})$

8. (a) $\frac{1}{\sqrt{2}}(1 + i)$ , $\frac{1}{\sqrt{2}}(-1 + i)$ , $\frac{1}{\sqrt{2}}(-1 - i)$ , $\frac{1}{\sqrt{2}}(1 - i)$ .

(b) $1$ , $-1$ , $\frac{-1 + i\sqrt{3}}{2}$ , $\frac{-1 - i\sqrt{3}}{2}$ , $\frac{1 - i\sqrt{3}}{2}$ , $\frac{1 + i\sqrt{3}}{2}$ .

(c) $\sqrt[3]{12}\,(\cos 80^\circ + i \sin 80^\circ)$ ,

$\sqrt[3]{12}\,(-\cos 20^\circ - i \sin 20^\circ)$ ,

$\sqrt[3]{12}\,(\cos 40^\circ - i \sin 40^\circ)$ .

9. $1.5094 + 1.3122\ i$ ,

$-1.3122 + 1.5094\ i$ ,

$-1.5094 - 1.3122\ i$ ,

$1.3122 - 1.5094\ i$ .

10. Let $\omega = \cos\frac{2\pi}{n} + i \sin\frac{2\pi}{n}$ . Then the $n^{th}$ roots of unity are $1, \omega, \omega^2, \ldots, \omega^{n-1}$ . Now $\omega$ satisfies the equation

$$z^n - 1 = (z - 1)(z^{n-1} + z^{n-2} + \ldots + z^2 + z + 1) = 0 ;$$

but $\omega \neq 1$ ; therefore

$$\omega^{n-1} + \omega^{n-2} + \ldots + \omega^2 + \omega + 1 = 0 .$$ Q.E.D.

*11. (a) The non-real $(n + 1)^{st}$ roots of unity. (See Exercise 8.)

(b) The additive inverses of the non-real $(n + 1)^{st}$ roots of unity.

Exercises 12-7. Answers.

1. $\omega_1 = \omega_2{}^2 = \omega_1{}^4$ is equivalent to $\omega_1 = \omega_1{}^4$, or, $1 = \omega_1{}^3$, $\omega_1 \neq 0$. Treatment for $\omega_2$ symmetric to $\omega_1$.

2. (a) $2\sqrt{3}(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6})$.

   (b) $2\sqrt{2}(\cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4})$.

   (c) $1(\cos 143^\circ + i \sin 143^\circ)$.

   (d) $1(\cos 124^\circ + i \sin 124^\circ)$.

   (e) $r[\cos(2\pi - \theta) + i \sin(2\pi - \theta)] = r[\cos(-\theta) + i \sin(-\theta)]$.

   (f) $7(\cos 25^\circ 35' + i \sin 25^\circ 35')$.

   (g) $1(\cos 182^\circ + i \sin 182^\circ)$.

   (h) $1.062(\cos 29^\circ 56' + i \sin 29^\circ 56')$.

3. (a) $(-1 + \sqrt{3}) + i(1 + \sqrt{3})$.

   (b) $0.4684 + 0.2852\, i$.

   (c) $0.4384 + 0.8988\, i$.

4. (a) Pair $\omega^k$ with $\omega^{n-k}$ in the product. Each such pair $= \omega^n = 1$. If $n$ is even, $n = 2m$ for some integer $m$. In the pairing, $\omega^m$ is to be paired with $\omega^{2m-m} = \omega^m$; but $\omega^m$ appears only once in the product and is the only unpaired factor; hence, the product $= \omega^m$.

   (b) In pairing as above, if $n$ is odd, there is no unpaired root and the product $= 1$.

5. Let $\arg z = \theta$ ; we know $|\omega| = |\omega^2| = 1$ and $\omega_2 = \omega_1^2$ .
From (12-3), $zz^1 = rr^1[\cos(\theta + \theta^1) + i \sin(\theta + \theta^1)]$ .
Polar forms of $z$ , $z\omega$, $z\omega^2$ are then

$$z = |z|[\cos\theta + i \sin\theta]$$
$$z\omega = |z|[\cos(\theta + \tfrac{2\pi}{3}) + i \sin(\theta + \tfrac{2\pi}{3})]$$
$$z\omega^2 = |z|[\cos(\theta + \tfrac{4\pi}{3}) + i \sin(\theta + \tfrac{4\pi}{3})] .$$

This shows the three rays to $z$ , $z\omega$ , $z\omega^2$ , separated by equal angles in multiples of $\frac{2\pi}{3}$ , intersecting circle of radius $|z|$ . Since equal central angles subtend equal arcs, the points form the inscribed equilateral triangle. Since $\omega_1 = \omega_2^2$ , we could have started with $\omega_2$ and obtained the same result.

6. Take successive $n^{th}$ roots of unity. For simplicity, take the distance from $\omega_0 = 1 + 0 \cdot i$ to

$$\omega_1 = \cos(\tfrac{2\pi}{n}) + i \sin(\tfrac{2\pi}{n}) .$$

$$d^1 = \sqrt{[1 - \cos(\tfrac{2\pi}{n})]^2 + (\sin\tfrac{2\pi}{n})^2} = \sqrt{2 - 2\cos(\tfrac{2\pi}{n})} = 2 \sin\tfrac{\pi}{n} .$$

Polygon inscribed in circle of radius $|z|$ is proportional and $d = 2|z|\sin\frac{\pi}{n}$ .

7. (a) $\omega_k = 8^{1/8}[\cos(\frac{\pi}{16} + \frac{k\pi}{2}) + i \sin(\frac{\pi}{16} + \frac{k\pi}{2})]$ , $k = 0, 1, 2, 3$.
(b) $x = 0$ , $-\frac{1}{2}\omega_k$ , where $\omega_k$ is a $5^{th}$ root of unity.

## 12-8. Suggested Test Items.

Part I: True-False.

1. The product of any two complex numbers is a complex number.
2. If the argument of $z$ is $\pi$, then $z$ is a real number.
3. If arg $z$ is $\frac{3\pi}{2}$, then arg$(z^2)$ is $3\pi$.
4. The product of the three cube roots of unity is $-1$.
5. The complex number $\cos\theta + i \sin\theta$ is the multiplicative identity in the complex number system.
6. If $z = |z|(\cos\theta + i \sin\theta)$, when $|z| \neq 0$, then the multiplicative inverse of $z$ is
$$\frac{1}{|z|}(\cos\theta + i \sin\theta).$$
7. In an Argand diagram the number $8 + 8i$ represents a point on the circle with center at the origin and radius $8$.
8. If $z \neq 0$ and $w$ is a cube root of $z$, then $(\frac{-1 - i\sqrt{3}}{2})w$ is also a cube root of $z$.
9. If $z_1 z_2$ represents a point on the circle with center at the origin and radius $1$ and if $z_1$ represents a point inside that circle, then $z_2$ represents a point outside the circle.
10. The roots of a quadratic equation with real coefficients are complex numbers.

Part II. Multiple Choice.

Directions: Select the response which best completes the statement or answers the question.

11. The absolute value of $4i$ is

A. $-16$.
B. $\pm\sqrt{16}$.
C. $4$.
D. $2$.
E. $-4$.

12. If $z = r(\cos\theta + i \sin\theta)$, then $z^2$ is equal to

A. $r^2(2\cos\theta + 2i \sin\theta)$.
B. $r^2(\cos 2\theta \pm i \sin 2\theta)$.
C. $2r(\cos 2\theta + i \sin 2\theta)$.
D. $r^2(\cos 2\theta + i \sin 2\theta)$.
E. $r^2(\cos\theta^2 + i \sin\theta^2)$.

13. If $z_1 = 2 + 3i$ and $z_2 = 2 - 3i$, then $z_1 \cdot z_2$ is equal to

A. $4 + 0 \cdot i$.
B. $\sqrt{13}(\cos 0^\circ + i \sin 0^\circ)$.
C. $\sqrt{5}(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2})$.
D. $4 - 9i$.
E. $13 \cos 0^\circ$.

14. If $w$ is one of the roots of the equation $w^5 = z$, where $z \neq 0$, then another root is

A. $w^2$.
B. $w^4$.
C. $w(\cos 72^\circ + i \sin 72^\circ)$.
D. $\cos 72^\circ + i \sin 72^\circ$.
E. $|w|(\cos 72^\circ + i \sin 72^\circ)$.

15. If $z = 1 - i\sqrt{3}$, then arg z is equal to

A. $120^\circ$.
B. $60^\circ$.
C. $300^\circ$.
D. $150^\circ$.
E. $330^\circ$.

16. Which of the following expressions is the polar form of $i^5$ ?

A. $0 + 1 \cdot i$.
B. $\cos \pi + i \sin \pi$.
C. $\cos 2\pi + i \sin 2\pi$.
D. $\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}$.
E. $\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2}$.

17. Which of the following is a cube root of $i$ ?

A. $i$.
B. $\cos 150^\circ + i \sin 150^\circ$.
C. $\frac{1}{\sqrt{2}}(\cos 45^\circ + i \sin 45^\circ)$.
D. $-i$.
E. Both B and D are correct.

18. The polar form of the complex number $2\sqrt{2} - 2\sqrt{2}\,i$ is

A. $2\sqrt{2}(\cos 45^\circ + i \sin 45^\circ)$.
B. $4(\cos 315^\circ + i \sin 315^\circ)$.
C. $2\sqrt{2}(\cos 315^\circ + i \sin 315^\circ)$.
D. $4(\cos 135^\circ + i \sin 135^\circ)$.
E. $16(\cos 315^\circ + i \sin 315^\circ)$.

19. If $z = \sqrt{2}(\cos 60^\circ + i \sin 60^\circ)$, then $z^4$ is equal to

A. $8(\cos 240^\circ + i \sin 240^\circ)$.

B. $4(\frac{1}{2} + \frac{i\sqrt{3}}{2})$.

C. $(\sqrt{2})^4 (4 \cos 60^\circ + 4i \sin 60^\circ)$.

D. $4(\cos 240^\circ + i \sin 240^\circ)$.

E. $2 - 2i\sqrt{3}$.

20. If $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$, $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$ and $z_3 = r_3(\cos \theta_3 + i \sin \theta_3)$, then $\frac{z_1 \cdot z_3}{z_2}$ is equal to

A. $\frac{r_1 r_3}{r_2} \cdot [\cos \frac{\theta_1 + \theta_3}{\theta_2} + i \sin \frac{\theta_1 + \theta_3}{\theta_2}]$.

B. $\frac{r_1 r_3}{r_2} [\cos(\theta_1 + \theta_3 - \theta_2) + i \sin(\theta_1 + \theta_3 - \theta_2)]$.

C. $r_1 r_3[\cos(\theta_1 + \theta_3) + i \sin(\theta_1 + \theta_3)] - r_2(\cos \theta_2 + i \sin \theta_2)$

D. $\frac{r_1 r_3}{r_2}[\cos \frac{\theta_1 \theta_3}{\theta_2} + i \sin \frac{\theta_1 \theta_3}{\theta_2}]$.

E. $\frac{r_1 + r_2}{r_3} [\cos \frac{\theta_1 + \theta_3}{\theta_2} + i \sin \frac{\theta_1 + \theta_3}{\theta_2}]$.

21. Which one of the following numbers is <u>not</u> a $10^{th}$ root of unity?

A. $-1$.

B. $\cos 10^\circ + i \sin 10^\circ$.

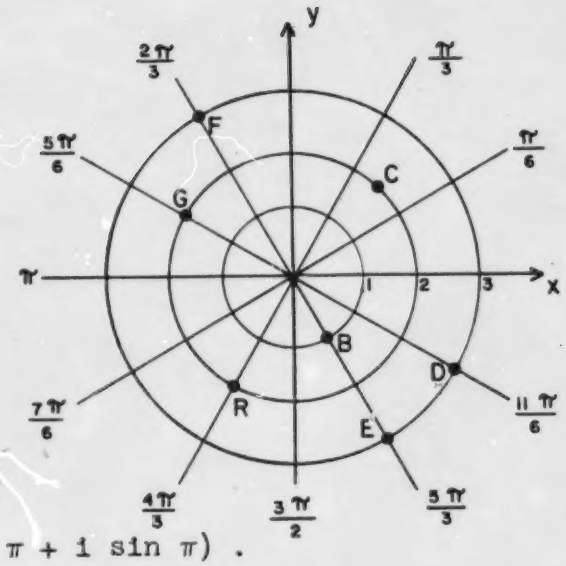
C. $\cos 108^\circ + i \sin 108^\circ$.

D. $1$.

E. $\cos 36^\circ + i \sin 36^\circ$.

Part III: Matching.

Directions: For each of the following questions choose the point on the diagram which represents the given complex number. Any choice may be used once, several times, or not at all.




22. $z = \sqrt{2} + i\sqrt{2}$ .

23. $z = 3(\cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3})$ .

24. $z = [\sqrt[4]{2}(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3})]^4$ .

25. $z = 3(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3})(\cos \pi + i \sin \pi)$ .

26. $z = \dfrac{6(\cos \frac{5\pi}{12} + i \sin \frac{5\pi}{12})}{3(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6})}$ .

Part IV: Problems.

27. Express the five fifth roots of $-i$ in polar form.

28. Solve the quadratic equation $z^2 - (2 + 4i)z + (4i - 3) = 0$ .

29. If $z = -\sqrt{3} + i$ , express $z^{12}$ in polar form.

30. If $z_1 = 2(\cos 25^\circ + i \sin 25^\circ)$ ,

$z_2 = 3(\cos 30^\circ + i \sin 30^\circ)$ ,

and $z_3 = 12(\cos 45^\circ + i \sin 45^\circ)$ ,

find $\dfrac{z_1^3 \cdot z_2^2}{z_3}$ .

Answers to Suggested Test Items.

I. 1. T
2. T
3. F
4. F
5. F
6. F
7. F
8. T
9. T
10. T

II. 11. C
12. D
13. E
14. C
15. C
16. D
17. E
18. B
19. D
20. B
21. B

III. 22. C
23. E
24. A
25. E
26. C

IV. 27. $\cos 54^\circ + i \sin 54^\circ$
$\cos 126^\circ + i \sin 126^\circ$
$\cos 198^\circ + i \sin 198^\circ$
$\cos 270^\circ + i \sin 270^\circ$
$\cos 342^\circ + i \sin 342^\circ$

28. $1 + 2i$ , $1 + 2i$

29. $2^{12} (\cos 1800^\circ + i \sin 1800^\circ)$
or $2^{12} (\cos 0^\circ + i \sin 0^\circ)$
or $2^{12}$
or $4096$ .

30. $6(\cos 90^\circ + i \sin 90^\circ)$
or $6i$ .

# Chapter 13

## SEQUENCES AND SERIES

### 13-1. Introduction.

This chapter treats various aspects of the subject of sequences and series. This section is a preliminary one which introduces definitions and some necessary notation for

finite sequence,
infinite sequence,
finite series,
infinite series, and
sum of a finite series of numbers.

The $\sum$ -notation for the sum of a series is introduced also. Finite and infinite sequences of numbers are ordered sequences of numbers of the forms

$$a_1, a_2, \ldots, a_n$$
$$a_1, a_2, \ldots, a_n, \ldots$$

respectively. There is a last number in the first line; but, as the three dots indicate, there is no last number in the second line. A finite sequence is formed by associating an element $a_k$ (which need not be a number) with each number $k$ in the set $\{1, 2, \ldots, k, \ldots, n\}$. In the same way, an infinite sequence is formed by associating an element $a_k$ with each number $k$ in the set $\{1, 2, \ldots, k, \ldots, n, \ldots\}$. Thus, a finite sequence is a function whose domain is $\{1, 2, \ldots, n\}$, and an infinite sequence is a function whose domain is $\{1, 2, \ldots, n, \ldots\}$.

An infinite series is defined to be the indicated sum

$$a_1 + a_2 + \ldots + a_n + \ldots$$

in this section. The teacher should examine also a definition

given in Section 13-5 of this Commentary.

It is important for the student to gain familiarity with the $\sum$-notation for finite and infinite series. This notation is standard throughout all branches of mathematics and in all parts of the world, and the student will continue to encounter it as long as he studies mathematics.

Probably the best advice that can be given for teaching this section is this: use lots of examples to help the student understand the new concepts and the new notation.

Exercises 13-1; Answers.

1. (a) $-1, -4, -7, -10, -13, -16, -19$

   (b) $\frac{3}{4}, \frac{6}{7}, \frac{9}{10}, \frac{12}{13}, \frac{15}{16}, \frac{18}{19}, \frac{21}{22}$

   (c) $\sqrt{2}, 2, 2\sqrt{2}, 4, 4\sqrt{2}, 8, 8\sqrt{2}$

   (d) $2\times5, 4\times10, 8\times20, 16\times40, 32\times80, 64\times160, 128\times320$

2. (a) $-3k + 2$

   (b) $\frac{3k}{3k+1}$

   (c) $2^{\frac{k}{2}}$

   (d) $2^k \times 5(2)^{k-1}$

3. (a) $\{-3k + 2\}_{k=1}^{7}$

   (b) $\{\frac{3k}{3k+1}\}_{k=1}^{7}$

   (c) $\{2^{\frac{k}{2}}\}_{k=1}^{7}$

   (d) $\{2^k \times 5(2)^{k-1}\}_{k=1}^{7}$

4. (a) $7 - 2 + 7 - 2 + 7 - 2 + 7$

   (b) $7 + 0 - 7 + 0 + 7 + 0 - 7$

   (c) $a + 2a + 3a + 4a + 5a + 6a + 7a$

   (d) $1 - 2 + 3 - 4 + 5 - 6 + 7$

5. (a) 22

   (b) 0

   (c) 28a

   (d) 4

6. (a) $\sum_{k=1}^{10} 2k - 3$ (c) $\sum_{k=1}^{\infty} k(k+2)$

(b) $\sum_{k=1}^{8} 2^k(-1)^{k+1}$ (d) $\sum_{k=1}^{\infty} 1 + (2-k)1$

7. (a) 1, 0, 0, 1, 3, 6, 10

(b) $1 + 0 - 1 - 512$

(c) $3 + 1 + 0 + 0 + 1 + 3 + 6$

(d) 16, -2, 1, -2, 16

8. (a), (b), (c) $-3 - 1 - \frac{1}{3} + 0 + \frac{1}{5} + \frac{1}{3}$

9. $(m+1)^2 - 14(m+1) = m^2 + 14m + 13$

10. $(-1)^{\frac{(k+2)(k+3)}{2}}(4k-3)$

11. These are simply the results of the corresponding expansions.

12. (a) no

(b) no

(c) yes

---

13-2. Arithmetic Sequences and Series.

This section treats an old topic, and it is thoroughly familiar to most teachers. It gives the necessary definitions concerning arithmetic sequences and series, and it derives formulas for the $n^{\text{th}}$ term of an arithmetic sequence and for the sum of $n$ terms.

The derivation of the formula for $S_n$ is not the one usually given. The traditional method is used to derive the formula

$$1 + 2 + \ldots + n = \frac{n^2 + n}{2},$$

and then this result is used to derive the formula for $S_n$ in

the general case. The traditional method can be used also as follows:

$$Sn = a_1 + [a_1 + d] + \ldots + [a_1 + (n - 2)d] + [a_1 + (n - 1)d]$$

$$Sn = [a_1 + (n - 1)d] + [a_1 + (n - 2)d] + \ldots + [a_1 + d] + a_1$$

$$2Sn = n[2a_1 + (n - 1)d]$$

$$Sn = \frac{n}{2}[2a_1 + (n - 1)d] = \frac{n}{2}(a_1 + an)$$

Exercises 13-2; Answers.

1. (a) arithmetic series, $d = 6$

   (b) not arithmetic

   (c) arithmetic series, $d = 5$

   (d) arithmetic series, $d = 4$

   (e) arithmetic series, $d = 4$

2. Series has 16 terms; $a_1 = -16$; $a_{16} = -1$

   $$S_{16} = \frac{16}{2}[-16 + (-1)] = -136$$

3. Series has 6 terms; $a_1 = -4$; $a_6 = 6$

   $$S_6 = \frac{6}{2}(-4 + 6) = 6$$

4. $a_3 = a_2 + d$

   $p = m + d \qquad d = p - m$

   $a_1 = a_2 - d = m - (p - m) = 2m - p$

   $a_4 = a_3 + d = 2p - m$

   $a_5 = a_4 + d = 2p - m + (p - m) = 3p - 2m$

5. $a_1 = \frac{3m - p}{2}$, $a_2 = m$, $a_3 = \frac{m + n}{2}$, $a_4 = p$.

6. $a_1 = -2$

7. $a_1 = 1$

8. (a) $a_{15} = 31$

(b) $a_{11} = 28$

(c) $a_9 = 34\frac{1}{2}$

9. $350 = 23 \cdot 15 + 5$

$\therefore$ Largest integer less than 350 and divisible by 23 is $23 \cdot 15 = 345$

smallest integer greater than 35 and divisible by 23 is $2 \cdot 3 = 46$

$\therefore$ There are 14 integers which are multiples of 23 between 35 and 350

$S_{14} = \frac{14}{2}(46 + 345) = 2{,}737$

10. (a) 35

(b) -2

(c) $5 + 2\sqrt{3}$

(d) $c^2 + cd$

11. Yes, the new common difference is 5d.

12. $d = \frac{5}{7}$ hence the six arithmetic means are:

$$\frac{59}{14}, \frac{69}{14}, \frac{79}{14}, \frac{89}{14}, \frac{99}{14}, \frac{109}{14}.$$

13. (a) 55

(b) 499,500

(c) -45

14. $\sum_{k=1}^{8} 6k = 216$

15. $\frac{n(n+1)}{2}a + nb$

16. $\frac{n^2 + n}{2} = 153$

$n^2 + n - 306 = 0$

$n = -18, \quad n = 17.$

$n = 17$ will check. The other result $n = -18$ will also check if one begins with the last term (17) and count backwards.

17. $\sum_{k=0}^{4} (ak + b) = 10a + 5b = 10$

$\sum_{k=1}^{4} (ak + b) = 10a + 4b = 14$

from which, $a = 3, \quad b = -4$

or alternately, from

$$\sum_{k=m}^{p} a_k = \sum_{k=m}^{n} a_k + \sum_{k=n+1}^{p} a_k \quad \text{for} \quad m < n < p,$$

$$\sum_{k=0}^{4} (ak + b) - \sum_{k=1}^{4} (ak + b) = \sum_{k=0}^{0} (ak + b) = b = 10 - 14 = -4$$

hence, from $10a + 5b = 10, \quad a = 3.$

18. $\frac{1}{2}(n + m + 1)(m - n)$

19. 579

20. $a_1 = a_n - (n - 1)d$

$S_n = na_n - \frac{n(n - 1)}{2}d$

21. From $3 - x - (-x) = (-x) - \sqrt{9 - 2x},$

$x^2 + 8x = x(x + 8) = 0$

$x = 0 \quad \text{or} \quad x = -8$

$x = 0$ does not satisfy. If $x = -8$, the series is 11, 8, 5.

22. Denote the numbers by $x + y$, $x$, $x - y$.
Then $x = -1$, $y = 3$ or $y = -3$.
For either value of $y$, we obtained the numbers
2, -1 and -4.

23. (a) There are 42 integers in the required sum,
$S_{42} = 60,501$.

(b) Our sequence begins with 7 and ends with 297 with
$d = 10$, and 30 integers in the sum.
$S_{30} = 4,560$.

---

13-3. Geometric Sequences and Series.

This section also treats a topic that is familiar to most teachers. It gives the necessary definitions concerning geometric sequences and series, and it derives formulas for the $n^{\underline{th}}$ term of a geometric series and for the sum of $n$ terms.

There is another derivation of the formula for $S_n$ that is frequently given.

$$Sn = a_1 + a_1r + a_1r^2 + \ldots + a_1r^{n-1}$$

$$rSn = \quad a_1r + a_1r^2 + \ldots + a_1r^{n-1} + a_1r^n$$

Subtract the second line from the first. Then

$$Sn - rSn = a_1 - a_1r^n$$

$$Sn(1 - r) = a_1(1 - r^n)$$

$$Sn = \frac{a_1(1 - r^n)}{1 - r} = \frac{a_1(r^n - 1)}{r - 1}$$

Exercises 13-3; Answers.

1. (a) -50, -250, -1250

(b) $-\frac{1}{2}$, $\frac{1}{3}$, $-\frac{2}{9}$

(c) $\frac{1}{7}$, $\frac{1}{49}$, $\frac{1}{343}$,

2. $b = \pm\sqrt{ac}$

3. (a) 1023

   (b) 61

   (c) 0

4. $\frac{r^5(1 - r^{94})}{1 - r}$

5. $r = 2$; $a_1 = 1$

   $63 = 2^n - 1$

   $n = 6$

6. 4

7. Yes!

   $1 + r + r^2 = 7$

   $(r + 3)(r - 2) = 0$

   $r = 2$ or $r = -3$

   The two series corresponding to these roots are:

   $1 + 2 + 4 = 7$

   $1 - 3 + 9 = 7$

8. $r = 1$; the number of terms in the sum is

   $n + m - n + 1 = m + 1$

   $Sm + 1 = m + 1$.

   $r \neq 1$; $S_{m+1} = \frac{r^n(r^{m+1} - 1)}{r - 1}$

9. $\frac{2^{2n+1} - 2}{3}$

10. (a) $k = \frac{2}{3}$

    (b) $x = -7$

11. 14

12. 1. $ar^2 = -4$

2. $ar^4 = -1$

1. $r^2 = -\frac{4}{a}$

2. $a \cdot \frac{16}{a^2} = -1$

$a = -16$

$r^2 = \frac{-4}{-16} = \frac{1}{4}$

$r = \frac{1}{2}$

$\therefore$ -16, -8

13. $a_1^3 r^3 = -216$; $a_1 r = -6$; $a_1 = -\frac{6}{r}$

the series is, $-\frac{6}{r}$, -6, -6r

and the sum of squares is

$$36\left(\frac{1}{r^2} + 1 + r^2\right) = 189$$

$$(4r^2 - 1)(r^2 - 4) = 0$$

$$r = \frac{1}{2}, \ -\frac{1}{2}, \ 2, \ -2.$$

$r = \frac{1}{2}$: -12, -6, -3

$r = -\frac{1}{2}$: 12, -6, 3

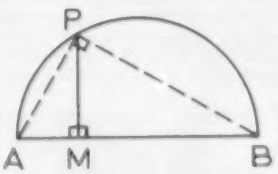
Series are repeated in reversed order for $r = 2$ and $r = -2$.

14. From plane geometry we know that angle APB is a right angle.

Therefore, AM:MP = MP:MB

or $\overline{MP}^2 = \overline{AM} \cdot \overline{MB}$

$\overline{MP} = \sqrt{\overline{AM} \cdot \overline{MB}}$




The definition of a geometric mean between any two numbers is that it is the square root of the product of the numbers.

Therefore, AM + MP + MB are the terms of a geometric series.

15. (a) 4, 16, 64

(b) $\sqrt[3]{25}$, $\sqrt[6]{5^5}$

(c) $4a^4b^2$, or $-4a^4b^2$

(d) $a_1 = a$; $an = b$; $n = 3$

$ar^2 = b$

$r^2 = \frac{b}{a}$

$r = \pm\sqrt{\frac{b}{a}}$; or $\pm\frac{\sqrt{ab}}{a}$; $a_2 = a_1 r = \pm\sqrt{ab}$

---

13-4. Limit of a Sequence.

The limit of a sequence is one of the really fundamental concepts in mathematics, and it is unfortunate that it is a difficult concept for most students.

Consider the nature of the limit of a sequence. Some sequences have limits, and some do not. Suppose that we agree to associate with each sequence that is convergent the limit of this sequence. It is clear that, according to the definition given in Chapter 3, this association defines a function. Let this function be denoted by L, and let the value of this function corresponding to the infinite sequence $a_1, a_2, \ldots, a_n, \ldots$ be denoted by

$$L(a_1, a_2, \ldots, a_n, \ldots).$$

The domain of this function consists of a proper subset of the all infinite sequences. The domain of L consists of the set of all convergent sequences (that is, the set of all sequences which have limits); the function L is not defined for most infinite sequences.

The function L has many interesting properties as follows.

(1) Every sequence of the form $a_1, a_2, \ldots, a_n, a, a, \ldots$ belongs to the domain of L, and

$$L(a_1, a_2, \ldots, a_n, a, a, \ldots) = a.$$

(2) If $a_1, a_2, \ldots, a_n, \ldots$ belongs to the domain of L, and if c is any number, then $ca_1, ca_2, \ldots, ca_n, \ldots$ also belongs to the domain of L, and

$$L(ca_1, ca_2, \ldots, ca_n, \ldots) = cL(a_1, a_2, \ldots, a_n, \ldots)$$

(3) If $a_1, a_2, \ldots, a_n, \ldots$ and $b_1, b_2, \ldots, b_n, \ldots$ belong to the domain of L, then $a_1 \pm b_1, a_2 \pm b_2, \ldots, a_n \pm b_n, \ldots$ and $a_1b_1, a_2b_2, \ldots, a_nb_n, \ldots$ also belong to the domain of L, and

$$L(a_1 \pm b_1, a_2 \pm b_2, \ldots, \pm an \pm bn, \ldots) = L(a_1, a_2, \ldots, a_n, \ldots) \pm L(b_1, b_2, \ldots, b_n, \ldots)$$

$$L(a_1b_1, a_2b_2, \ldots, a_nb_n, \ldots) = L(a_1, a_2, \ldots, a_n, \ldots) \cdot L(b_1, b_2, \ldots, b_n, \ldots).$$

Furthermore, if $bn \neq 0$ for all n and $L(b_1, b_2, \ldots, b_n, \ldots) \neq 0$, then the infinite sequence $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_n}{b_n}, \ldots$ also belongs to the domain of L, and

$$L\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_n}{b_n}, \ldots\right) = \frac{L(a_1, a_2, \ldots, a_n, \ldots)}{L(b_1, b_2, \ldots, b_n, \ldots)}.$$

It should now be pointed out that the properties listed in (1), (2), (3) are exactly the results stated in Theorems 13-4a 13-4b of the text; the notation is different, but the meaning is the same.

The limit of a sequence is thus the value of a special function. The limit function is another important example of a function whose domain is not a set of real numbers; it may be in order to recall that the domains of the trigonometric functions are the set of geometric objects called signed angles.

The usual treatment of limits focuses attention on how the number $L(a_1, a_2, \ldots, a_n, \ldots)$ is associated with the sequence $a_1, a_2, \ldots, a_n, \ldots$ . This part of the treatment is left on an intuitive basis in this section, and the teacher should present it intuitively also.

Exercises 13-4; Answers.

1. (a) $\frac{1}{2}$ (c) $\frac{1}{2}$

   (b) 1 *(d) $\frac{3}{2}$

2. The convergent series with their limits are:

   (b) 0 (f) 2

   (c) 0 (h) $\frac{2}{3}$

   (e) 1 (i) 3

3. (a) $\frac{3}{2}$ (c) $\frac{3}{5}$

   (b) 1 (d) 0

   (e) $\frac{8}{3}$

4. (a) See example (13-4e).

   (b) $\lim_{n\to\infty}(\frac{1}{n} - \frac{2}{n^2}) = \lim_{n\to\infty}\frac{1}{n} - \lim_{n\to\infty}\frac{2}{n^2}$ [13-4b(2)]

   $= \lim_{n\to\infty}\frac{1}{n} - 2\lim_{n\to\infty}\frac{1}{n^2}$ [13-4b(1)]

   $= 0 - 2\cdot 0 = 0$

5. $\lim \frac{2n^2 - 3n}{4n + 1} \longrightarrow \frac{2 - \frac{3}{n}}{\frac{4}{n} + \frac{1}{n^2}} \longrightarrow \frac{2}{0}$

6. $\lim \frac{2n^3}{5n - 1} \longrightarrow \frac{2}{\frac{5}{n^3} - \frac{1}{n^3}} \longrightarrow \frac{2}{0}$

7. $\frac{a + \frac{b}{n} + \frac{c}{n^2}}{d + \frac{e}{n} + \frac{f}{n^2}} \longrightarrow \frac{a}{d}$

Exercise 3(a) has a limit of $\frac{3n}{2n} = \frac{3}{2}$

Exercise 3(b) has a limit of $\frac{n^2}{n^2} = 1$

Exercise 3(c) has a limit of $\frac{3n^3}{5n^3} = \frac{3}{5}$.

8. If $d = 0$ and $a = 0$ and $e \neq 0$,

the $\lim \frac{an^2 + bn + c}{dn^2 + en + f} \longrightarrow \frac{b}{e}$

If $a = b = d = e = 0$, the limit $\longrightarrow \frac{c}{f}$ $(f \neq 0)$.

*9. Dividing each term of the numerator and denominator of

$\frac{a_0 n^r + a_1 n^{r-1} + \ldots + a_r}{b_0 n^r + b_1 n^{r-1} + \ldots + b_r}$ we get $\frac{a_0 + \frac{a_1}{n} + \ldots + \frac{a_r}{n^r}}{b_0 + \frac{b_1}{n} + \ldots + \frac{b_r}{n^r}} = \frac{a_0}{b_0}$

where $b \neq 0$

10. $\lim_{n \to \infty} \frac{1 - r^n}{1 - r} = \frac{\lim 1 - r^n}{\lim 1 - r} = \frac{1}{1 - r}$

11. $\lim_{n \to \infty} \frac{(n^2 - 3n + 5)(3n^3 - 1)}{n(n^4 - 17n + 11)} = \lim \frac{(1 - \frac{3}{n} + \frac{5}{n^2})(3 - \frac{1}{n^3})}{(1 - \frac{17}{n^3} + \frac{11}{n^4})} = 3$

12. (a) 1. [See Theorem 13-4a]

(b) 0

(c) 7. [13 - 4b (1) and 13-4a]

---

13-5. Sum of an Infinite Series.

In Section 13-1 an infinite series was defined to be the indicated sum

$$13\text{-}5a \qquad a_1 + a_2 + \ldots + a_n + \ldots, \quad \text{or} \quad \sum_{k=1}^{\infty} a_n.$$

From the modern point of view this definition is not sufficiently concrete to be satisfactory. It is more customary now to say that the infinite series denoted by the two symbols in 13-5a is the infinite sequence

$$13\text{-}5b \qquad s_1, s_2, \ldots, s_n, \ldots$$

where

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 = s_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 = s_2 + a_3 \\ &\vdots \\ s_n &= a_1 + a_2 + \ldots + a_n = s_{n-1} + a_n \\ &\vdots \end{aligned}$$

The infinite series denoted by the symbols in 13-5a is thus the sequence $(s_1, s_2, \ldots, s_n, \ldots)$ of partial sums $s_n$.

The sum of the infinite series 13-5a is

$$13\text{-}5c \qquad \lim_{n \to \infty} s_n$$

We write

$$13\text{-}5d \qquad \sum_{k=1}^{\infty} a_n = A,$$

and by this statement we mean nothing more nor less than

$$13\text{-}5e \qquad \lim_{n \to \infty} s_n = A$$

The symbol $\sum_{k=1}^{\infty} a_k$ is used to denote both the infinite series and its sum.

Let us consider the nature of the sum of an infinite series. With some infinite sequences $a_1, a_2, \ldots, a_n, \ldots$ we associate the number $A$ defined by 13-5e. It is clear that this association defines another function; we shall denote it by $S$. The value of this function corresponding to the sequence $a_1, a_2, \ldots, a_n, \ldots$ is $S(a_1, a_2, \ldots, a_n, \ldots)$.

The limit of a sequence is a value of the $L$ function discussed in Section 13-4 of this Commentary. We thus have the following set of relations:

$$13\text{-}5f \qquad \sum_{k=1}^{\infty} a_n = A$$

$$= \lim_{n \to \infty} s_n$$

$$= S(a_1, a_2, \ldots, a_n, \ldots)$$

$$= L(s_1, s_2, \ldots, s_n, \ldots).$$

It follows that properties of the sum of an infinite series can be obtained from properties of the $L$ function. From (1) in Section 13-4 of this Commentary, we have

$$S(a_1, a_2, \ldots, a_n, 0, 0, \ldots) = L(s_1, s_2, \ldots, s_n, s_n, s_n, \ldots)$$

$$= s_n$$

$$= a_1 + a_2 + \ldots + a_n$$

As a result of this property, we can say that the sum of an infinite series is a generalization of the concept of sum of a finite series.

From (2) in Section 13-4 of this Commentary, we obtain another property of the S function. If $a_1, a_2, \ldots, a_n, \ldots$ belongs to the domain of S, than $ca_1, ca_2, \ldots, ca_n, \ldots$ also belongs to the domain of S, and

$$S(ca_1, ca_2, \ldots, ca_n, \ldots) = cS(a_1, a_2, \ldots, a_n, \ldots)$$

The proof can be given as follows. If $s_n$ is the $n^{\underline{th}}$ partial sum of $\sum_{k=1}^{\infty} a_k$, then $cs_n$ is the $n^{\underline{th}}$ partial sum of $\sum_{k=1}^{\infty} ca_k$,

$$\begin{aligned} S(ca_1, ca_2, \ldots, ca_n, \ldots) &= L(cs_1, cs_2, \ldots, cs_n, \ldots) \\ &= cL(s_1, s_2, \ldots, s_n, \ldots) \\ &= cS(a_1, a_2, \ldots, a_n, \ldots) \end{aligned}$$

in the same way, we can use (3) in Section 13-4 of this Commentary to show that, if $a_1, a_2, \ldots, a_n, \ldots$ and $b_1, b_2, \ldots, b_n, \ldots$ belong to the domain of S, then $a_1 \pm b_1$, $a_2 \pm b_2$, $\ldots$, $a_n \pm b_n$, $\ldots$ belongs to the domain of S, and

$$\begin{aligned} &S(a_1 \pm b_1, a_2 \pm b_2, \ldots, a_n \pm b_n, \ldots) \\ &\quad = S(a_1, a_2, \ldots, a_n, \ldots) \pm S(b_1, b_2, \ldots, b_n, \ldots) \end{aligned}$$

There are two basic questions in connection with the study of infinite series. (a) Does the infinite series $a_1 + a_2 + \ldots + a_n + \ldots$ converge? (b) If the infinite series converges, what is its sum? It will be in order to comment on these two questions. Question (a) can be stated in many different ways, in fact, the following questions are all really different ways of asking the same question.

(1) Does the infinite series $\sum_{k=1}^{\infty} a_k$ converge?

(2) Does $\lim_{n \to \infty} s_n$ exist?

(3) Does the infinite sequence $a_1, a_2, \ldots, a_n, \ldots$ belong to the domain of definition of the function S?

(4) Does the infinite sequence $s_1, s_2, \ldots, s_n, \ldots$ belong to the domain of definition of the function L?

More advanced courses in mathematics contain many tests which enable us to tell whether a given infinite series $\sum_{k=1}^{\infty} a_n$ converges or diverges. Many of these tests are applied directly to the given infinite sequence $a_1, a_2, \ldots, a_n, \ldots$ rather than to the derived infinite sequence $s_1, s_2, \ldots, s_n, \ldots$ .

Question (b) in the last paragraph is a completely separate question in the following sense: it may be known that a given infinite series $\sum_{k=1}^{\infty} a_k$ converges, but it may still be impossible to find its sum. From the definition of a limit, however, it follows that $s_n$ is a good approximation to the sum of the infinite series if $n$ is sufficiently large. This remark forms the basis for the use of infinite series for calculation of the type involved in the computation of tables of trigonometic functions, exponential functions, logarithm functions, and so forth.

Exercises 13-5; Answers.

1. Solution: $s_1 = 2$

$s_2 = 9$

$s_3 = 21$

$s_4 = 38$

Series Arithmetic: find $s_n$ by theorem 13-2a

$$s_n = \frac{n}{2}[2 + 2 + (n - 1)5]$$

$$= \frac{n}{2}(5n - 1)$$

2. Solution: $a_1 = \frac{7}{100}$; $a_2 = \frac{7}{100}\cdot\frac{1}{10}$; $a_3 = \frac{7}{100}\cdot(\frac{1}{10})^2$

$s_1 = \frac{7}{100}$

$s_2 = \frac{77}{1000}$

$s_3 = \frac{777}{10,000}$

$$s_n = \frac{\frac{7}{100}\left(1 - (\frac{1}{10})^n\right)}{1 - \frac{1}{10}} \qquad \text{(theorem 13-3a)}$$

$$s_n = \frac{7}{10^{n+1}}\cdot\frac{10^n - 1}{10 - 1}$$

3. Solution: Series is arithmetic ∴ by [theorem 13-2a]

$$s_n = \frac{3n}{2}(11 - n)$$

4. Solution: Series is geometric

$a = \frac{1}{2}$; $r = \frac{3}{2}$,

$s_n = (\frac{3}{2})^n - 1$ by theorem 13-3a

5. Solution: $a_1 = s_1 = 2$

$$\therefore\ 2 + 4 + 6 + 8 + \ldots + 2n + \ldots = \sum_{n=1}^{\infty} 2n$$

$a_2 = s_2 - a_1 = 6 - 2 = 4$

$a_3 = s_3 - s_2 = 12 - 6 = 6$

$a_4 = 20 - 12 = 8$

$a_5 = 30 - 20 = 10$

$a_n = 2n$

6. Solution: $a_1 = s_1 = 2$; $a_2 = 6 - 2 = 4$; $a_3 = 14 - 6 = 8$

$a_4 = 30 - 14 = 16$; $a_5 = 62 - 30 = 32$

2, 4, 8, 16, 36, ..., ____

$2 + 2^2 + 2^3 + 2^4 + 2^5 + \ldots + 2^n + \ldots = \sum_{n=1}^{\infty} 2^n$

7. Solution:

$$a_1 = s_1 = 2 \qquad = 2 = -(-2)$$
$$a_2 = s_2 - s_1 = -4 \qquad = -(2)^2 = -(-2)^2$$
$$a_3 = s_3 - s_2 = 8 \qquad = (2)^3 = -(-2)^3$$
$$a_4 = s_4 - s_3 = -16 \qquad = -(2)^4 = -(-2)^4$$
$$a_5 = s_5 - s_4 = 32 \qquad = 2^5 = -(-2)^5$$
$$a_n \qquad = -(-2)^n$$

$$2 - 4 + 8 - 16 + \ldots + -(-2)^n + \ldots = \sum_{n=1}^{\infty} -(-2)^n$$

8. Solution:

$$a_1 = 3 \qquad = 2(1) + 1$$
$$a_2 = 8 - 3 = 5 \qquad = 2(2) + 1$$
$$a_3 = 15 - 8 = 7 \qquad = 2(3) + 1$$
$$a_4 = 24 - 15 = 9 \qquad = 2(4) + 1$$
$$a_n = 2n + 1$$

$$3 + 5 + 7 + \ldots + (2n + 1) + \ldots = \sum_{n=1}^{\infty} (2n + 1)$$

9. Solution:

$$a_1 = 2 \qquad = 2.1 \qquad a_n = n(n + 1)$$
$$a_2 = 8 - 2 = 6 \qquad = 2.3$$
$$a_3 = 20 - 8 = 12 \qquad = 3.4$$
$$a_4 = 40 - 20 = 20 \qquad = 4.5$$
$$a_n = n(n + 1)$$

$$2 + 6 + 12 + \ldots + n(n + 1) + \ldots = \sum_{n=1}^{\infty} n(n + 1)$$

10. $a_n = \frac{3}{(3n+2)(3n-1)} = \frac{1}{3n-1} - \frac{1}{3n+2}$

Thus $a_1 = \frac{1}{2} - \frac{1}{5}$ ; $a_2 = \frac{1}{5} - \frac{1}{8}$ ; $a_3 = \frac{1}{8} - \frac{1}{11}$ ; ... ;

$$a_n = \frac{1}{3n-1} - \frac{1}{3n+2}$$

$$S_n = a_1 + a_2 + a_3 + \ldots + a_n = \frac{1}{2} - \frac{1}{3n+2}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(\frac{1}{2} - \frac{1}{3n+2}\right) = \frac{1}{2} .$$

11. Evidently $s_k = a_k + s_{k-1}$ or $a_k = s_k - s_{k-1}$

Hence $a_k = \frac{k}{2(3k+2)} - \frac{k-1}{2[3(k-1)+2]} = \frac{k}{2(3k+2)} - \frac{k-1}{2(3k-1)}$

$$= \frac{1}{(3k+2)(3k-1)} .$$

Substituting $k = 1, 2, 3, \ldots$ we obtain the series

$$\frac{1}{10} + \frac{1}{40} + \frac{1}{88} + \ldots + \frac{1}{(3k+2)(3k-1)} + \ldots$$

12. Solution: $\left(\frac{2}{4} - 2\right) + \left(\frac{2}{9} - \frac{2}{4}\right) + \left(\frac{2}{16} - \frac{2}{9}\right) + \ldots$

$$s_1 = \frac{2}{4} - 2 = -2 + \frac{2}{4}$$

$$s_2 = -2 + \frac{2}{9}$$

$$s_3 = -2 + \frac{2}{16}$$

$$s_n = -2 + \frac{2}{(n+1)^2}$$

$$\therefore \sum_{k=1}^{\infty} \left(\frac{2}{k^2 + 2_k + 1} - \frac{2}{k^2}\right) = \lim_{n\to\infty} \left(\frac{2}{(n+1)^2} - 2\right) = -2$$

[See: 13-4b(2)]

13. Proof: $(\frac{2^2}{4} - \frac{1}{3}) + (\frac{3^2}{5} - \frac{2^2}{4}) + (\frac{4^2}{6} - \frac{3^2}{5}) + \ldots$

$$s_1 = -\frac{1}{3} + \frac{2^2}{4}$$

$$s_2 = -\frac{1}{3} + \frac{3^2}{5}$$

$$s_3 = -\frac{1}{3} + \frac{4^2}{6}$$

$$s_n = -\frac{1}{3} + \frac{(n+1)^2}{n+3}$$

$$\lim_{n\to\infty} \left(-\frac{1}{3} + \frac{n^2 + 2n + 1}{n + 3}\right) = -\frac{1}{3} + \lim_{n\to\infty} \frac{1 + \frac{2}{n} + \frac{1}{n^2}}{\frac{1}{n} + \frac{3}{n^2}}$$

$\therefore$ Series diverges since $\lim_{n\to\infty} s_n$ does not exist.

14. Proof:

$$s_1 = 1$$

$$s_2 = -1$$

$$s_3 = 2$$

$$s_4 = -2$$

If $n$ is even

$$s_n = -\frac{n}{2}$$

and if $n$ is odd

$$s_n = \frac{n+1}{2}$$

in either case $s_n$ has no limit and therefore the series diverges.

15. $\lim_{n\to\infty} \frac{1}{n^2} \sum_{k=0}^{n-1} k = \lim_{n\to\infty} \frac{1}{n^2} \cdot \frac{n(n-1)}{2}$

$$= \frac{1}{2} \lim_{n\to\infty} \frac{n^2 - n}{n^2} = \frac{1}{2} \lim_{n\to\infty} (1 - \frac{1}{n})$$

$$= \frac{1}{2}.$$

16. $\lim_{n\to\infty} \frac{1}{n^3} \sum_{k=0}^{n-1} k^2 = \lim_{n\to\infty} \frac{1}{3} \cdot \frac{(n-1)n(2n-1)}{6}$

$= \frac{1}{6} \lim_{n\to\infty} (1 - \frac{1}{n})(2 - \frac{1}{n})$

$= \frac{1}{3}.$

17. $s_n = \sum_{k=1}^{n} (\frac{1}{2k+1} - \frac{1}{2k+3}) = (\frac{1}{3} - \frac{1}{5}) + (\frac{1}{5} - \frac{1}{7}) + \ldots$

$+ (\frac{1}{2n+1} - \frac{1}{2n+3}).$

$= \frac{1}{3} - \frac{1}{2n+3}.$

$\lim_{n\to\infty} s_n = \lim_{n\to\infty} (\frac{1}{3} - \frac{1}{2n+3}) = \frac{1}{3}$

18. $e^{\frac{1}{2}} = 1 + \frac{1}{2} + \frac{1}{2\cdot 4} + \frac{1}{6\cdot 8} + \frac{1}{24\cdot 16} + \ldots$

$= 1.00000$
$\quad .50000$
$\quad .12500$
$\quad .02083$
$\quad .00260$
$\quad .00026$
$\quad .00002$
$\approx 1.64871$

19. $\log_e 1.1 = .10000 - .00500$
$\quad .00033 - .00003 + \ldots$
$= .10033 - .00503 + \ldots$
$\approx 0.0953$

---

13-6. The Infinite Geometric Series.

The infinite geometric series has many important applications in mathematics, and it provides especially useful illustrations of the ideas developed in Section 13-5. Given the infinite geometric series

13-6a $$a_1 + a_1r + \ldots + a_1r^{n-1} + \ldots$$

or

13-6a $$\sum_{k=1}^{\infty} a_1r^{k-1},$$

it is possible to determine precisely the values of $r$ for which the series converges and for which the series diverges. Furthermore, it is possible to find the sum of the series for every value of $r$ for which the series converges. The value of $s_n$ is obtained by using the formula in Section 13-3 for the sum of a finite geometric series. We obtain $s_n$ in a form such that its limit can be evaluated directly. Thus, in this special case, we show that the series converges and find the sum of the series simultaneously. There are only a few types of infinite series for which this simple situation exists.

Exercises 13-6; Answers.

1. (a) $r = \frac{1}{2}$; $s = \dfrac{1}{1 - \frac{1}{2}} = 2$

(b) $r = -\frac{1}{3}$; $s = 9[\dfrac{1}{1 - (-\frac{1}{3})}] = \frac{27}{4}$

2. (a) $r = r$; $s = \dfrac{r}{1 - r}$

(b) $r = (1 - a)$; $s = \dfrac{1}{1 - a} \cdot \dfrac{1}{1 - (1 - a)} = \dfrac{1}{a - a^2}$

$|1 - a| < 1$

$0 < a < 2$

3. (a) $0.\overline{5}$

$a = .5; \quad 5 = .1$

$$s_n = \frac{a}{1 - r} = \frac{.5}{1 - .1} = \frac{.5}{.9} = \frac{5}{9}$$

(b) $0.\overline{062}$

$a = .062; \quad r = .01$

$$s_n = \frac{a}{1 - r} = \frac{.062}{1 - .01} = \frac{.062}{.99} = \frac{62}{990} = \frac{31}{495}$$

(c) $3.\overline{297}$

$a = .297; \quad r = .001$

$$s_n = \frac{.297}{1 - .001} = \frac{.297}{.999} = \frac{297}{999} = \frac{33}{111}$$

Therefore common fraction = $3\frac{33}{111}$

(d) $2.6\overline{9}$

$a = .09; \quad r = .1$

$$s_n = \frac{.09}{1 - .1} = \frac{.09}{.9} = \frac{9}{90} = \frac{1}{10}$$

Therefore, common fraction is $2\frac{7}{10}$.

(It should be noted here that $2.6\overline{9}$ ... is another way of writing $2.7\overline{0}$ ... This should be explained to the student to avoid confusion on this.)

4. $a = 72"; \quad r = .9; \quad s = \frac{72}{1 - .9} = 720$ inches.

5. $$\frac{2}{3} = \frac{1}{1 - x}$$
$$2 - 2x = 3$$
$$2x = -1$$
$$x = \frac{-1}{2}$$

6. $$\frac{1 + x}{x} = \frac{x}{1 - x}$$
$$x^2 = 1 - x^2$$
$$2x^2 = 1$$
$$x^2 = \frac{1}{2}$$
$$x = \pm\sqrt{\frac{1}{2}}$$

7. 1. $\frac{a}{1 - r} = \frac{3}{2}$

2. $\frac{a}{1 + r} = \frac{3}{4}$

1. $2a + 3r = 3$

2. $\underline{4a - 3r = 3}$

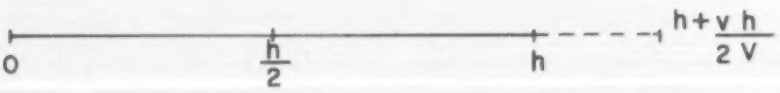
$6a = 6$

$a = 1$ Therefore, $r = \frac{1}{3}$

8. $a_1 = 12$; $r = \frac{1}{2}$; $s = \frac{12}{1 - \frac{1}{2}} = 24"$.

9. Set handicap $h = 5000$ yards. To cover half this distance the hare will need $h/2V$ minutes. Meanwhile the tortoise goes $vh/2V$ yards, and at the end of the time interval required to traverse this distance the contestants will still be apart by

$$(h + \frac{vh}{2V}) - \frac{h}{2} = \frac{h(V + v)}{2V}$$




Note that the new distance apart is obtained from the old by multiplying by $(V + v)/2V$. We now repeat the process replacing $h$ by $h(V + v)/2V$. The second interval of time will be

$$\frac{h(V + v)}{(2V)^2}$$

and the distance apart at the end of that interval will be

$$\frac{h(V + v)^2}{(2V)^2}.$$

True, there will be infinitely many time intervals, but the sum of these will be

$$T = \frac{h}{2V}\left[1 + \frac{V+v}{2V} + \left(\frac{V+v}{2V}\right)^2 + \ldots\right]$$

$$= \frac{h}{2V} \cdot \frac{1}{1 - \frac{V+v}{2V}} = \frac{h}{V-v}.$$

That is, the hare <u>will</u> overtake the tortoise in $\frac{5000}{999} = 5.005$ min. The fallacy was in using the word "never" to describe a sum of infinitely many time intervals, when that sum was 5.005. We can check our result by the equation

$$TV = h + Tv.$$

The left-hand member represents the distance from the starting point the hare will be in T minutes, the right-hand member represents the corresponding distance for the tortoise. This is known as one of Zeno's paradoxes. Reference may be made to this and two others, The Arrow and The Achilles, in Dantzig's <u>Numbers, the Language of Science</u>.

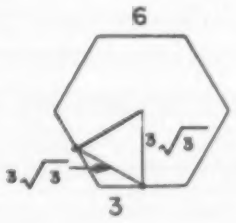
10. The perimeters are of lengths

36, $18\sqrt{3}$, 27, ...

$a_1 = 36$; $r = \frac{1}{2}\sqrt{3}$;

$s = 72(2 + \sqrt{3})$

$\approx 268.7$ inches




11. (a) $a_1 = 12$; $r = \frac{1}{3}\sqrt{5}$; $s = 9(3 + \sqrt{5})$

$\approx 47.12$ inches.

(b) $a_1 = 9$; $r = \frac{5}{9}$; $S = 20\frac{1}{4}$ sq. in.

## Answers to Miscellaneous Exercises

1. $\sum_{k=0}^{3} [(-2)^k - 2k] = \sum_{k=0}^{3} (-2)^k - 2 \sum_{k=0}^{3} k$

$= -15 - 12$

$= -17$

2. (a) $8 + 4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \ldots$

(b) $3 + 6 + 9 + 12 + 15 + 18 + \ldots$

(c) $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \frac{1}{1024} + \frac{1}{4096} + \ldots$

(d) $\frac{1}{3} - 1 + 3 - 9 + 27 - 81 + \ldots$

(e) $a^2 + a^4 + a^6 + a^8 + a^{10} + a^{12} + \ldots$

(f) $1 - 1 - 1 + 1 + 1 - 1 + \ldots$

3. 20

4. $1 + 2 + 3 + \ldots + n = \frac{(n + 1)n}{2}$

Writing several terms we get:

$\frac{1}{2}[1 \cdot 2 - 0] + \frac{1}{2}[2 \cdot 3 - 2 \cdot 1] + \frac{1}{2}[3 \cdot 4 - 2 \cdot 3] + \ldots$

$+ \frac{1}{2}[n(n + 1) - n(n - 1)]$

Which becomes $\frac{n(n + 1)}{2}$

5. For $k = 1, 2, \ldots$ the identity is

$3 = 2^2 - 1^2, \quad 5 = 3^2 - 2^2, \ldots$

Hence the series is

$1 + (3^2 - 1^2) + (5^2 - 3^2) + \ldots + (n^2 - [n - 1]^2).$

This series is collapsible. When parentheses are removed each term cancels except $n^2$.

6. Writing:

$$\frac{1}{2} - \frac{1}{3} = \frac{1}{2\cdot 3}$$

$$\frac{1}{3} - \frac{1}{4} = \frac{1}{3\cdot 4}$$

$$\frac{1}{4} - \frac{1}{5} = \frac{1}{4\cdot 5}$$

$$\begin{array}{ccc} \cdot & \cdot & \cdot \\ \cdot & \cdot = & \cdot \\ \cdot & \cdot & \cdot \end{array}$$

$$\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$$

$$\therefore 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

7. Take 6 and 24 as any two positive integers. The required ratio r is determined by

$$a_1 r^2 = a_3$$

$$6r^2 = 24$$

$$r = \pm 2,$$

and the geometric mean is $a_1 r = \pm 12$

Arithmetic mean is $\frac{6 + 24}{2} = 15$

Note that the geometric mean is smaller than the arithmetic mean.

Now take a and b as any two positive integers. The required r is determined by

$$ar^2 = b$$

$$r = \pm \frac{\sqrt{ab}}{a},$$

and the geometric mean is $ar = \pm \sqrt{ab}$.

Arithmetic mean is $\frac{a + b}{2}$.

$\sqrt{ab} \leq \frac{a + b}{2}$, equality obtaining when $a = b$.

To show $\sqrt{ab} \leq \frac{a+b}{2}$, observe that for any real numbers a, b.

$$(a-b)^2 \geq 0$$
$$(a-b)^2 + 4ab \geq 4ab$$
$$(a+b)^2 \geq 4ab$$
$$a+b \geq 2\sqrt{ab}.$$

8. $\frac{2}{1} = \frac{1}{2}(\frac{1}{1} + \frac{3}{1})$; $2 = \frac{4}{2}$ or $\frac{1}{2} = \frac{2(1)(\frac{1}{3})}{\frac{1}{3}+1} = \frac{2}{3} \div \frac{4}{3} = \frac{2}{3}\cdot\frac{3}{4} = \frac{1}{2}$

$\frac{3}{1} = \frac{1}{2}(\frac{2}{1} + \frac{4}{1})$; $3 = \frac{6}{2}$ or similar to above.

9. Number of vertices of a cube is 8.
Number of faces of a cube is 6.
Number of edges of a cube is 12
8 is the harmonic mean between 6 and 12
because $8 = \frac{2(6)(12)}{6+12}$

$$8 = \frac{144}{18}$$
$$8 = 8$$

10. Try the result for 2 and 8 first and then generalize:
Let a and b be the two numbers.
Then the geometric mean is $\sqrt{ab}$
the arithmetic mean is $\frac{a+b}{2}$
the harmonic mean is $\frac{2ab}{b+a}$

To Prove: $\sqrt{ab} = \sqrt{\frac{a+b}{2} \div \frac{b+a}{2ab}}$

$$\sqrt{ab} = \sqrt{\frac{a+b}{2} \cdot \frac{2ab}{b+a}}$$
$$\sqrt{ab} = \sqrt{ab}$$

11. $\sum_{k=0}^{5} \frac{1}{(-3)^k(2k+1)} = 1 - \frac{1}{3\cdot 3} + \frac{1}{5\cdot 3^2} - \frac{1}{7\cdot 3^3} + \frac{1}{9\cdot 3^4} - \frac{1}{11\cdot 3^5}$

$= 1.000 - .111$
$\quad .022 - .005$
$\quad .001 - .000$
$= 1.023 - .116 = 0.907$
$S = 3.464\ (0.907)$
$\approx 3.14$

The limit of the sum is $\pi$.

12. 3,250 yards.

13. $\frac{3n(3n+5)}{8}$

14. If $n$ = number or days, then $10n$ is the distance travelled.

$$a_1 = 8, \quad a_n = 8 + \tfrac{1}{2}(n-1)$$
$$10n = \tfrac{n}{2}(16 + \tfrac{n-1}{2})$$
$$n = 9$$

15. $\sum_{k=1}^{n} (2k-1) = n^2 = 12321$
$n = 111$

16. 436

17. (a) 850

(b) 825. This series is obtained from the series for (a) by subtracting 1 from each of the 25 terms.

18. $a_1 = \$44$; $a_{144} = \$86.90$

19. If the series is $a - d$, $a$, $a + d$, $a + 2d$,

(a) $(a-d)^2 + (a+2d)^2 > a^2 + (a+d)^2$
expanding and subtracting,
$4d^2 > 0$,

(b) $(a-d)(a+2d) < a(a+d)$
$-2d^2 < 0$,

20. (a) Yes. The new difference is still d and the new sum is $s_n + nc$. This is an application of

$$\sum_{k=1}^{n} (a_k + c) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} c$$

$$= s_n + nc.$$

(b) Yes. The new difference is cd and new sum is c times the original sum.

$$\sum_{k=1}^{n} ca_k = c \sum_{k=1}^{n} a_k = cs_n.$$

(c) No.

(d) No.

(e) Yes. New ratios is $\frac{1}{r}$.

(f) Yes. New ratios is still r.

21. $n + 1$

22. $\frac{1}{b + c} - \frac{1}{c + a} = \frac{1}{c + a} - \frac{1}{a + b}$

$$\frac{a + b}{(b + c)(c + a)} = \frac{b - c}{(c + a)(a + b)}$$

$$\frac{b - c}{b + c} = \frac{b - c}{a + b}$$

$$a^2 - b^2 = b^2 - c^2$$

$\therefore$ $a^2$, $b^2$, $c^2$ are in arithmetic progression.

23. This is an arithmetic progression,

$$\frac{1 - \sqrt{x}}{1 - x} + \frac{1}{1 - x} + \frac{1 + \sqrt{x}}{1 - x} + \ldots$$

with $d = \frac{\sqrt{x}}{1 - x}$.

$$S_n = \frac{n}{1 + \sqrt{x}} + \frac{n(n - 1)\sqrt{x}}{2(1 - x)}.$$

24. $na_1 + \frac{n(n-1)d}{2} = ma_1 + \frac{m(m-1)d}{2}$

$$a_1 = \frac{d}{2(n-m)}[m(m-1) - n(n-1)]$$

$$= \frac{d}{2(n-m)}(m-n)(m+n-1)$$

$$= -\frac{d}{2}(m+n-1)$$

$$S_{m+n} = (m+n)a_1 + \frac{(m+n)(m+n-1)d}{2}$$

$$= (m+n)[a_1 + \frac{(m+n-1)d}{2}]$$

$$= (m+n)\frac{d}{2}[-(m+n-1) + (m+n-1)]$$

$$= 0.$$

*25. $\frac{m}{2}(a_1 + a_m) = n$

$\frac{n}{2}(a_1 + a_n) = m$

$a_1 + a_m = \frac{2n}{m}$; $a_1 + a_n = \frac{2m}{n}$

$2a_1 + (m-1)d = \frac{2n}{m}$; $2a_1 + (n-1) = \frac{2m}{n}$

from which $d = -\frac{2(m+n)}{mn}$

$$a_1 = \frac{m^2 + n^2 + mn - m - n}{mn}$$

$$S_{m+n} = \frac{m+n}{2}[\frac{2n^2 + 2m^2 + 2mn - 2m - 2n}{mn} - \frac{(m+n-1)2(m+n)}{mn}]$$

$$= \frac{m+n}{2} \cdot \frac{-2mn}{mn} = -(m+n).$$

## Illustrative Test Questions

Note: Teachers may prefer to change some of the multiple choice items (e.g. 30, 31, 33, 35, 36, 39) as problem questions.

A. Multiple Choice.

Directions: Select the response which best completes the statement or answers the question. Cross out the letter of your choice on the answer sheet.

1. A sequence of numbers is best described as
   (a) A set of numbers such that the difference between successive numbers is constant.
   (b) A correspondence which associates one number with each natural number $n$.
   (c) A set of numbers in ascending order of magnitude.
   (d) A set of numbers with commas between successive numbers.
   (e) An indicated sum of a set of numbers.

2. The third term of the sequence $\left\{r^{2k}\right\}_{k=2}^{10}$ is
   (a) $4^{2k}$.
   (b) $r^{6}$.
   (c) $r^{8}$.
   (d) $3^{2k}$
   (e) $4^{8}$.

3. Which one of the following symbols represents an infinite sequence of numbers $s_n$?
   (a) $\left\{s_k\right\}_{k=5}^{\infty}$.
   (b) $s_1, s_2, s_3, \ldots s_n$.
   (c) $\sum_{k=1}^{\infty} s_k$.
   (d) $s_\infty$.
   (e) $\left\{s_k\right\}_{k=1}^{n}$.

4. The symbol $\left\{\frac{k}{k+2}\right\}_{k=-1}^{2}$ is equivalent to which of the following sequences?

(a) $-1, 0, 1, 2.$

(b) $-1, 0, \frac{1}{3}, \frac{1}{2}.$

(c) $\frac{1}{3}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}.$

(d) $0, \frac{1}{3}, \frac{1}{2}.$

(e) $0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}.$

5. The $k^{th}$ term in the series $1\cdot3 + 2\cdot5 + 3\cdot7 + 4\cdot9 + \ldots$ could be

(a) $k(2k - 1)$.

(b) $k(k + 2)$.

(c) $k(2k + 1)$.

(d) $2k(2k + 1)$.

(e) $7k - 4$.

6. The series $0\cdot3 + 2\cdot5 + 4\cdot7 + \ldots$ is equivalent to

(a) $\sum_{k=0}^{\infty} 2k(k + 3)$

(b) $\sum_{k=2}^{\infty} 2k(2k + 3)$

(c) $\sum_{k=1}^{\infty} (3k - 3)(3k)$.

(d) $\sum_{k=1}^{\infty} (k - 1)(k + 2)$.

(e) $\sum_{k=2}^{\infty} (k - 2)(k + 1)$.

7. The series $1\cdot3 + 3\cdot5 + 5\cdot7 + 7\cdot9$ is equivalent to

(a) $\sum_{k=0}^{3} (k + 1)(k + 3)$.

(b) $\sum_{k=1}^{4} k(k + 2)$.

(c) $\sum_{k=1}^{4} (2k - 1)(2k + 1)$.

(d) $\sum_{j=2}^{6} (j - 1)(j + 1)$.

(e) $\sum_{k=1}^{4} 3(4k - 3)$.

8. What is the $18^{th}$ term of the arithmetic series $5 + 1 + (-3) + \ldots$ ?

(a) -63.
(b) -67.
(c) -71.
(d) 77.
(e) 73.

9. If the first term of an arithmetic progression is -3 and the difference between any two successive terms is 2, then the $n^{th}$ term is

(a) $n(n - 4)$.
(b) $2n - 3$.
(c) $2n - 5$.
(d) $-3 + 6n$.
(e) $5 - 3n$.

10. A formation has 24 men in the first row, 23 in the second, and so on to the last row, which has only one man. How many men are there in the formation?

(a) 276.
(b) 288.
(c) 300.
(d) 576.
(e) 375.

11. $\sum_{k=1}^{10} (2k + 1) =$

(a) 120.
(b) 231.
(c) 110.
(d) 99.
(e) 21.

12. If a, b, c is an arithmetic progression, then $a - b + c =$

(a) a.
(b) b.
(c) c.
(d) $a + c$.
(e) $-b$.

13. The seventh term of the sequence 2, -4, 8, ... is

(a) -10.
(b) -128.
(c) -256.
(d) $\frac{1}{32}$
(e) -34.

14. What is the fourth term of the geometric sequence $\sqrt{3}$, 3, ...?

(a) 6.
(b) 9.
(c) $6\sqrt{3}$.
(d) $9\sqrt{3}$.
(e) 27.

15. A man gives his son an allowance of 1 cent on the first day of the month, and each day thereafter, the allowance is twice the amount given the preceding day. Which of the following expressions is equal to the total number of cents that the boy receives in a month that has 30 days?

(a) $\frac{2^{30} - 1}{2 - 1}$.
(b) $\frac{2^{29} - 1}{2 - 1}$.
(c) $\frac{2^{30} + 1}{2 + 1}$.
(d) $1 \cdot 2^{29}$.
(e) $2(2^{30} - 1)$.

16. For what value of n does $\sum_{k=1}^{n} 5 \cdot 2^{k-1} = 315$?

(a) 8.
(b) 5.
(c) 6.
(d) 32.
(e) 7.

17. $\sum_{k=1}^{20} 2^k =$

(a) $\frac{2(2^{20} - 1)}{2 - 1}$.

(b) $\frac{2(2^{19} - 1)}{2 - 1}$

(c) $10(2^{20} + 2)$.

(d) $2^{20} - 2$.

(e) None of the above is correct.

18. The positive geometric mean between 3 and 54 is

(a) 27.

(b) $6\sqrt{3}$.

(c) $9\sqrt{2}$.

(d) $9\sqrt{3}$.

(e) 162.

19. An infinite series has a sum if

(a) The partial sums do not exceed a fixed number M.

(b) The partial sums remain finite as n increases.

(c) The sequence of partial sums has a limit.

(d) The $n^{th}$ term of the series approaches zero as n increases.

(e) The partial sums alternate in sign and decrease in absolute value.

20. Which of the following infinite series has the numbers $\frac{1}{8}$, $\frac{4}{8}$, and $\frac{9}{8}$ as the first three terms in its sequence of partial sums?

(a) $\frac{1}{8} + \frac{4}{8} + \frac{9}{8} + \ldots$ .

(b) $\frac{1}{8} + \frac{3}{8} + \frac{5}{8} + \ldots$ .

(c) $(\frac{1}{4} - \frac{1}{8}) + (\frac{1}{2} - \frac{1}{4}) + (1 - \frac{15}{16}) + \ldots$ .

(d) $\sum_{k=1}^{\infty} \frac{k^2}{8}$ .

(e) $\sum_{k=1}^{\infty} \frac{2k - 1}{4k + 4}$ .

21. Which of the following series is divergent?

(a) $2 + 1 + \frac{1}{2} + \ldots$ .

(b) $1 + \frac{1}{8} + \frac{1}{64} + \ldots$ .

(c) $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \ldots$ .

(d) $\frac{0.01}{1} + \frac{0.01}{2} + \frac{0.01}{3} + \ldots$ .

(e) $.3 + .03 + .003 + \ldots$ .

22. $\lim_{n \to \infty} \frac{n^2 + 2n - 3}{n^2 - 1} =$

(a) 0.

(b) 1.

(c) 2.

(d) 3.

(e) The limit does not exist.

23. $\lim_{k \to \infty} \frac{5k - 3}{5 + 2k} =$

(a) $\frac{5}{2}$

(b) 1.

(c) $-\frac{3}{2}$ .

(d) $-\frac{3}{5}$ .

(e) The limit does not exist.

24. $\lim_{n \to \infty} \frac{3n^2}{2n - 5} =$

(a) $\frac{3}{2}$ .

(b) $-\frac{3}{5}$ .

(c) -1.

(d) 0.

(e) The limit does not exist.

25. Find the sum of the geometric series $\sum_{k=0}^{\infty} ar^k$ when $a = 1$ and $r = -\frac{1}{2}$.

(a) 2.

(b) $\frac{3}{2}$ .

(c) $\frac{1}{2}$ .

(d) $\frac{2}{3}$ .

(e) The series has no sum.

26. The repeating decimal $.\overline{23}$ is equivalent to the geometric series whose common ratio $r$ and first term $a$ have the values

(a) $r = .01$, $a = 23$

(b) $r = .1$, $a = .23$

(c) $r = .23$, $a = .01$

(d) $r = .01$, $a = .23$

(e) $r = .0001$, $a = 23$

27. What is the sum of the odd-numbered terms of the geometric progression $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \ldots$ ?

(a) $\frac{1}{2}$.

(b) $\frac{3}{4}$.

(c) $\frac{5}{4}$.

(d) $\frac{4}{3}$.

(e) 2.

28. The sum of all numbers of the form $2k + 1$, where $k$ takes on integral values from 1 to $n$ is

(a) $n^2$.

(b) $n(n + 1)$.

(c) $n(n + 2)$.

(d) $(n + 1)^2$.

(e) $(n + 1)(n + 2)$.

29. The sum of the squares of the first $n$ positive integers is given by the expression $\frac{n(n + c)(2n + k)}{6}$, if $c$ and $k$ are respectively

(a) 1 and 2.

(b) 3 and 5.

(c) 2 and 2.

(d) 1 and 1.

(e) 2 and 1.

30. Two men set out at the same time to walk towards each other from M and N, 72 miles apart. The first man walks at the rate of 4 mph. The second man walks 2 miles the first hour, $2\frac{1}{2}$ miles the second hour, 3 miles the third hour, and so on in the arithmetic progression. Then the men will meet.

(a) in 7 hours.
(b) in $8\frac{1}{4}$ hours.
(c) nearer M than N.
(d) nearer N than M.
(e) midway between M and N.

31. By adding the same constant to each of 20. 50, 100, a geometric progression results. The common ratio is

(a) $\frac{5}{3}$.
(b) $\frac{4}{3}$.
(c) $\frac{3}{2}$.
(d) $\frac{1}{2}$.
(e) $\frac{1}{3}$.

32. The arithmetic mean (average) of a set of 50 numbers is 38. If two numbers, namely, 45 and 55, are discarded, the mean of the remaining set numbers is

(a) 36.5
(b) 37
(c) 37.2
(d) 37.5
(e) 37.52

33. A harmonic progression is a sequence of numbers such that their reciprocals are in arithmetic progression.

Let $s_n$ represent the sum of the first n terms of the harmonic progression; for example, $s_3$ represents the sum of the first three terms. If the first three terms of a harmonic progression are 3, 4, 6, then

(a) $s_4 = 20$.
(b) $s_4 = 25$.
(c) $s_5 = 49$.
(d) $s_6 = 49$.
(e) $s_2 = \frac{1}{2}s_4$.

34. When simplified the product $(1 - \frac{1}{3})(1 - \frac{1}{4})(1 - \frac{1}{5})\ldots(1 - \frac{1}{n})$ becomes

(a) $\frac{1}{n}$.

(b) $\frac{2}{n}$.

(c) $\frac{2(n - 1)}{n}$.

(d) $\frac{2}{n(n + 1)}$.

(e) $\frac{3}{n(n + 1)}$.

35. Let $s_9$ be the sum of the first nine terms of the sequence, $x + a$, $x^2 + 2a$, $x^3 + 3a$, ... . Then $s_9$ equals

(a) $\frac{50a + x + x^8}{x + 1}$.

(b) $50a - \frac{x + x^{10}}{x - 1}$.

(c) $\frac{x^9 - 1}{x + 1} + 45a$.

(d) $\frac{x^{10} - x}{x - 1} + 45a$.

(e) $\frac{x^{11} - x}{x - 1} + 45a$.

36. For the infinite series

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} - \frac{1}{32} + \frac{1}{64} - \frac{1}{128} - \ldots$$

let s be the sum. Then s equals

(a) 0.

(b) $\frac{2}{7}$.

(c) $\frac{6}{7}$.

(d) $\frac{9}{32}$.

(e) $\frac{27}{32}$.

37. The arithmetic mean between $\frac{x + a}{x}$ and $\frac{x - a}{x}$, when $x \neq 0$, is (the symbol $\neq$ means "not equal to")

(a) 2, if $a \neq 0$.

(b) 1.

(c) 1, if $a = 0$ only.

(d) $\frac{a}{x}$.

(e) x.

38. A 16-quart radiator is filled with water. Four quarts are removed and replaced with pure antifreeze liquid. Then four quarts of the mixture are removed and replaced with pure antifreeze. This is done a third and a fourth time. The fractional part of the final mixture that is water is

(a) $\frac{1}{4}$.

(b) $\frac{81}{256}$.

(c) $\frac{27}{64}$

(d) $\frac{37}{64}$.

(e) $\frac{175}{256}$.

39. The first term of an arithmetic series of consecutive integers is $k^2 + 1$. The sum of $2k + 1$ terms of this series may be expressed as

(a) $k^3 + (k + 1)^3$.

(b) $(k - 1)^3 + k^3$.

(c) $(k + 1)^3$.

(d) $(k + 1)^2$.

(e) $(2k + 1)(k + 1)^2$.

Part II. Problems

40. Find the number of terms in the geometric series $5 + 10 + 20 + \ldots$ whose sum is 1275.

41. Change the decimal $2.\overline{35}$ to a common fraction.

42. If $1 + 3 + 5 + \ldots + k = 121$, $k = ?$

43. In a puzzle contest 1480 dollars in prize money is divided among the 8 leading contestants. The money is divided according to the following scheme: 10 dollars is paid to the lowest ranking prize winner and each of the other 7 receives a fixed amount more than the preceding person. How much does the leading money winner receive?

44. Insert 4 positive terms between 9 and 288 to form a geometric sequence.

45. The number of bacteria in milk doubles every 3 hours. If there are n bacteria at a given time, how many will there be at the end of 24 hours from then?

46. Each operation of a vacuum pump removes $\frac{1}{4}$ of the air remaining in a cylinder. How much of the air present before the first operation remains in the cylinder after the sixth operation?

## Answers to Suggested Test Items

Part I.

| | | |
|---|---|---|
| 1. B | 14. B | 27. D |
| 2. C | 15. A | 28. C |
| 3. A | 16. C | 29. D |
| 4. B | 17. A | 30. E |
| 5. C | 18. C | 31. A |
| 6. A | 19. C | 32. D |
| 7. C | 20. B | 33. B |
| 8. A | 21. D | 34. B |
| 9. C | 22. B | 35. D |
| 10. C | 23. A | 36. B |
| 11. A | 24. E | 37. B |
| 12. B | 25. D | 38. B |
| 13. B | 26. D | 39. A |

Part II.

40. 8

41. $2\frac{35}{99}$

42. 21

43. $360

44. 18, 36, 72, 144

45. $2^8n$

46. $(\frac{3}{4})^6 \approx 0.178$ of tank.

Chapter 14

# PERMUTATIONS, COMBINATIONS, and THE BINOMIAL THEOREM

## 14-1. Introduction, Counting Problems.

In the introductory section we talk in a general way about the counting process and mention some of the kinds of enumeration problems arising in various fields of knowledge.

We list three important ideas used in counting collections of "things". We shall exploit these ideas repeatedly throughout the chapter. It may be observed here that in this chapter we use only the non-negative integers for all of our problems since our problems are problems of counting. The things we count may be "objects" of any kind, including various kinds of numbers. Thus, in Example 14-2c we count rational fractions (we use integers, of course, in order to count them). In Section 14-5 we discuss the binomial theorem- a theorem about expressions which may represent any kind of numbers, real or complex, (or the elements of any commutative ring)- but our method is a counting method. The coefficients which arise from our count are integers, although x and y themselves may represent numbers of any kind.

In a sense, when we come to this chapter, we turn our backs on the real and complex number systems and work solely with non-negative integers. Thus, there is no logical reason why one could not skip directly from Section 1-3 or 1-5, Chapter 1, to Chapter 14.

Most of Section 14-1 is devoted to indicating some of the kinds of counting problems one may meet in various fields. The talk about the large numbers which arise in some of them is intended to bolster our case for the necessity of deriving general results to solve, in one blow, classes of problems. We point out that, although many specific problems may be solved by a routine enumeration of cases, this method is, (i) very often thoroughly impractical because of the extremely large number of cases, (ii) not at all in line with our object, which is to develop a theory capable of handling classes of problems of this sort.

In order not to be misunderstood, we must admit quite candidly that there are very many counting problems which the methods we present - and all other known methods - are quite incapable of handling. In cases where direct enumeration is out of the question-because of the brevity of human life or the nature of the problem-we are totally unable to find any answers. It seems to be a fact of human nature - without which mathematics and science might not exist - that man has a talent for posing more questions than he can answer.

We illustrate our three fundamental ideas by recasting the proof given in Chapter 13 for

$$1 + 2 + \ldots + n = \frac{n(n + 1)}{2}$$

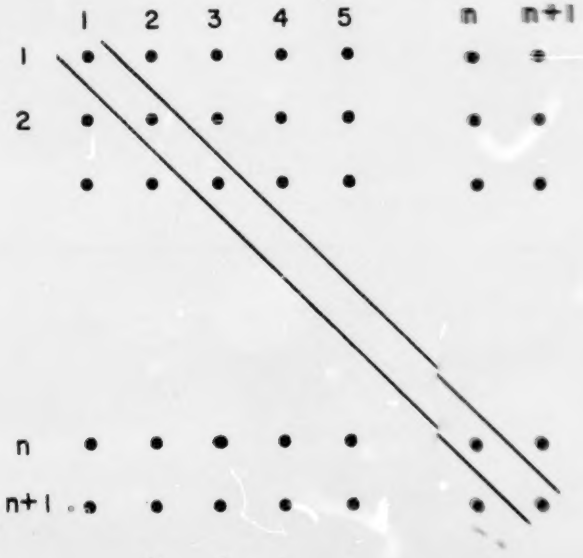
in terms of an array of dots. That our "new" proof is merely a recasting of the "old" one may be seen by adding the number of dots in each row

$$\begin{array}{l} 1 + n \\ 2 + (n - 1) \\ \hline\hline n + 1 \end{array}$$

and multiplying by the number of rows.

How we "draw the line" through our array of dots may cause a little uneasiness with some students who will worry about how we know where it "comes out" at the bottom. Note that, in row 1, it is above the first dot, below the second; in the second row it is above the second, below the third; ...; in the $n^{th}$ row, above the $n^{th}$ dot, below the $(n + 1)^{st}$.

A variant of this proof may be found interesting. Suppose we take a square array of $n + 1$ dots in each of $n + 1$ rows, isolating those dots on the "diagonal".




There are $n + 1$ dots on the diagonal (one in each row, one in each column). Below the diagonal, the second row has 1 dot, the third row has 2 dots,..... the $(n + 1)^{st}$ row has $n$ dots. Thus, below the diagonal are $1 + 2 + \ldots + n$ dots. Above the diagonal, there are also $1 + 2 + \ldots + n$ dots (replace the word "row" by "column" above). The sum of the number of dots in these three parts is

$$(n + 1) + 2s,$$

where $s = 1 + 2 + \ldots + n$. But we have $n + 1$ rows each with $n + 1$ dots. Therefore,

$$\begin{aligned} (n + 1) + 2s &= (n + 1)^2 \\ 2s &= (n + 1)^2 - (n + 1) \\ &= (n + 1)n \\ s &= \frac{n(n + 1)}{2} \end{aligned}$$

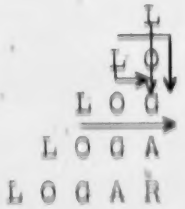
This alternative proof is a bit more complicated since it divides the array into three parts instead of two, but perhaps it is easier to see where the lines of demarcation "come out".

Almost all of our reasoning in later sections of this chapter is based on counting the elements in some rectangular array. It is for this reason that we present the proof for $1 + 2 + \ldots + n = \frac{n(n+1)}{2}$. We want to get the student started thinking this way.

There are two exercises at the end of Section 1-1. The first exercise presents a popular puzzle whose solution is a counting-problem in our sense. We give a big hint designed to lead the student to form a conjecture which will solve it. Making a rigorous proof for the conjecture - and for nearly every one of our results in later sections - requires the method of "mathematical induction" or some variant of it. (Cf., This Commentary, Section 1-3). We present none of these induction proofs, although we ought to for the sake of logical completeness. If we did present such proofs, we would have a much longer chapter and a more difficult one. We have tried - in each problem we consider - to carry through the cases for enough to reveal the general pattern and hope we have gone far enough to engender "reasonable" confidence in our generalizations. We maintain here that we can justify this confidence with induction proofs, and hope the reader will accept our word that we can.

Exercises 14-1. Answers

1. $2^9 = 512$. The first five rows have the number of "spelling paths" tabulated as follows. The arrows in the diagram show the paths involved in the first three rows:




$1 = 2^0$
$2 = 2^1$
$4 = 2^2$
$8 = 2^3$
$16 = 2^4$

Students may count the paths from each L and develop the Pascal Triangle pattern (see Exercises 14-4, Part 23), but it is probably best to shelve this discussion until then.

2. (a)

| + | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 6 | 7 | 8 | 9 | 10 | 11 | 12 |

(b) (i) the 7; appearing six times in the table.

(ii) Multiples of 3; appearing twelve times in the table as against nine powers of 2.

(iii) Non-primes; appearing twenty one times in the table as against fourteen primes.

---

14-2. Ordered Multiples.

The ideas in this section are fundamental for all the work in the later sections of this chapter.

This section is concerned with the cardinality of cartesian products of sets, although we omit mention of these words in the text. (The cardinality of a set is the number of elements in it.)

The cartesian product of two sets A, B (in that order) is the set of ordered couples of the form (a,b) where a is an element of A and b is an element of B; it is denoted by $A \times B$. An example with which the student is already familiar is the cartesian product $R \times R$, where R is the set of real numbers. Its elements are the ordered pairs (x,y) where x and y are real numbers. Coordinate geometry was invented in 1637 by

Rene DesCartes who, associating the two sets R,R with perpendicular number-lines, introduced these ordered couples as representatives of the points in the plane determined by the number lines. The term "cartesian" is used to honor the man who first did this.

For our counting problems we are interested only in the case in which the sets A and B are finite. If A and B have, respectively, n and m elements, then there are $n \times m$ elements in the cartesian product. This is our fundamental use of $I_3$, the third of our "counting ideas" in Section 14-1. It is stated as a formal principle in the text in Section 14-2. In the same section it is later extended to a more general form involving an arbitrary - but finite - number of finite sets.

We introduce ordered triples (a,b,c) as ordered couples of the form ((a,b),c) by "dropping" the inner pair of parentheses. Thus, we take as our set of ordered triples the cartesian product $(A \times B) \times C$, and our dropping of the parentheses corresponds to dropping the parentheses in the cartesian product $(A \times B) \times C$. But the cartesian product is not associative! Thus, we appear to be cheating. Our defense is the following: there is a one-to-one correspondence between the set of couples ((a,b),c) and the triples (a,b,c) when the latter are suitably defined otherwise, so that - for our counting problems - we may use the one to represent the other. Our offense is therefore, merely an "absue of the language". We prefer our tower of couples because we feel it is easier to see how to count triples, quadruples, etc., when we look at them this way.

We step up to ordered quadruples, quintuples, etc., and say we can do it in general for ordered m-triples. (This assertion conceals one of our tacit inductions.)

R. D. Luce, Some Basic Mathematical Concepts, SMSG Studies in Mathematics, Volume I, has an introductory discussion of cartesian products and other ideas of set theory. See also Elementary Mathematics of Sets with Applications, Mathematical Association of America.

The exercises are designed to develop practice in associating ordered m-triples with the items we want to count in various problems. This is practice in using fundamental idea number one of Section 1-1. Teachers will find it necessary to assign only as many of these exercises as the students require to develop confidence in their ability to handle them.

Exercises 14-2. Answers

1. $12(5) = 60$
2. $4(2) = 8$
3. $8(11) = 88$
4. $12(8) = 96$
5. $6(2) = 12$
6. $4(3) = 12$
7. $9(9) = 81$. Note that a digit may be repeated as in 55.
8. $10(10) = 100$. Note that one poster could win both prizes.
9. $4^2 = 16$
10. $26^2 = 676$
11. $4^3 = 64$.
12. $5(9^2) = 405$
13. $4.6^2 = 144$
14. $4^2(6^2) = 576$
15. $4(6)(3)(2) = 144$
16. $12(3)(120) = 4320$
17. $4^4 = 256$
18. $12(9)(10) = 1080$
19. $9(10^4) = 90,000$
20. $2^{10} = 1024$

*21. (a) $4(5^2) = 100$

(b) $4^2(3) = 48$

*22. $5(8) - 1(2) = 38$

*23. $5(8) - 2(2) - 1(2) = 34$

14-3. Permutations.

We introduce permutations as ordered m-triples, without duplication of elements of a given set which has at least m elements. Our problem is to count them. We do this by considering in turn small values of m (specifically m = 2 and m = 3). Using the technique developed in Section 14-2, we set up the procedure for stepping to larger values of m and announce (induction!) the general formula. We give some illustrations and then exercises in which the student is to count some permutations on his own.

Exercises 14-3. Answers.

1. $7^5 = 16,807$; $P(7,5) = 2,520$
2. $P(25,3) = 13,800$
3. $P(6,3) = 120$; $6^3 = 216$
4. $P(6,4) = 360$; $6^4 = 1296$
5. $P(4,4) = 24$; $4^4 = 256$
6. $8^7 = 2,097,152$; $P(8,7) = 40320$
7. $7! = 5040$
8. $P(6,4) = 360$
9. $P(4,3) = 24$
10. $P(20,3) = 6840$
11. $P(3,3) = 6$
12. $P(4,4) = 24$
13. $P(4,2) = 12$
14. $5 \cdot P(6,4) = 1800$
15. $P(4,4) \cdot P(5,5) = 2880$
16. $P(26,2)(10^4) = 6 \times 10^6$
17. $7! = 5040$

18. 6

19. $8! = 40{,}320$

20. (a) 11

(b) 336

(c) 2730

(d) 1326

(e) 2

(f) 455

(g) 60

(h) $\frac{1}{60}$

(i) 720

21. (a) $(n - 2)[n(n - 1) - 930] = 0$

$n^2 - n - 930 = 0$

$n = 31$; $n \neq 30$, since $2 < n$

(b) $n(n - 1)(n - 2)\,[(n - 3)(n - 2) - 20\,] = 0$

$n^2 - 7n - 8 = 0$

$n = 8$; $n \neq -1$, since $5 \leq n$

(c) $n(n - 1)[(n + 2)(n + 1) - 72] = 0$

$n^2 + 3n - 70 = 0$

$n = 10$; $n \neq 0$, $n \neq 1$, $n \neq -7$, Since $2 \leq n$.

(d) $(n - 1)[n(n + 1) - 10(n - 1)] = 0$

$n^2 - 9n + 20 = 0$

$n = 4$ or $5$.

22. (a) $(n + 3)(n + 2)(n + 1)$, or $P(n + 3,3)$

(b) $n(n - 1)(n - 2)$, or $P(n,3)$

(c) $n(n + 1)(n + 2)$, or $P(n + 2,3)$

(d) $\frac{1}{n(n-1)}$

(e) $(n - m)!$

(f) $(n - 1)!\, n(n + 1) = (n + 1)!$

(g) $\frac{n! + (n - 1)!}{n!\,(n - 1)!} = \frac{(n - 1)!\,(n + 1)}{n!\,(n - 1)!} = \frac{n + 1}{n!}$

(h) $\frac{(n + 1)!}{n![(n + 1) - 1]} = \frac{n + 1}{n}$

(i) $\frac{(n - 1)!\,(n + 1)\,[n(n + 1) + 1]}{(n - 1)!\,(n + 1)} = (n + 1)^2$

23 (a) $P(n,3) + 3.P(n,2) + P(n,1) =$

$$n(n - 1)(n - 2) + 3n(n - 1) + n$$
$$= n^3 - 3n^2 + 2n + 3n^2 - 3n + n$$
$$= n^3$$

(b) $(n + 1)\,[n \cdot n! + (2n - 1)(n - 1)! + (n - 1)(n - 2)!] =$

$$(n - 2)!\,(n + 1)[n^2(n-1) + (2n-1)(n-1)+(n-1)]$$
$$= (n-2)!(n+1)(n-1)[n^2 + (2n - 1) + 1]$$
$$= (n - 2)!\,(n-1)(n+1)(n^2 + 2n)$$
$$= (n - 2)!\,(n - 1)\,n\,(n + 1)\,(n + 2)$$
$$= (n + 2)!$$

(c) $P(n + 1, m) = (n + 1)\, n(n - 1)(n - 2) \ldots [(n+1)-m+1]$

$$= (n + 1)\, n(n - 1)(n - 2) \ldots (n - m + 2)$$
$$= (n + 1)[n(n - 1)(n - 2) \ldots [n - (m-1)+1]$$
$$= (n + 1)\, P(n,m - 1)$$

(d) $m \cdot P(n-1,m-1) + P(n-1, m) =$

$$m[(n-1)(n-2)\ldots[(n-1)-(m-1)+1]] +$$

$$[(n-1)(n-2)\ldots[(n-1)-m+1]]$$

$$= m[(n-1)(n-2)\ldots(n-m+1)] +$$

$$[(n-1)(n-2)\ldots(n-m)]$$

$$= m[(n-1)(n-2)\ldots(n-m+1)] +$$

$$[(n-1)(n-2)\ldots(n-m+1)(n-m)]$$

$$= (n-1)(n-2)\ldots(n-m+1)[m+(n-m)]$$

$$= n(n-1)(n-2)\ldots(n-m+1)$$

$$= P(n,m)$$

Note that it is easier to start with the right member and derive from it the left member of the equality. The symmetric property of the equals relation discussed in Chapter 1 completes the proof if one wishes to be more precise.

(e) $P(n-2,m) + 2m \cdot P(n-2, m-1) + m(m-1) \cdot P(n-2,m-2)$

$$= (n-2)(n-3)(n-4)\ldots[(n-2)-m+1]$$

$$+ 2m(n-2)(n-3)\ldots[(n-2)-(m-1)+1]$$

$$+ m(m-1)(n-2)(n-3)\ldots[(n-2)-(m-2)+1]$$

$$= (n-2)(n-3)\ldots(n-m-1)$$

$$+ 2m(n-2)(n-3)\ldots(n-m)$$

$$+ m(m-1)(n-2)(n-3)\ldots(n-m+1)$$

$$= (n-2)(n-3)\ldots(n-m+1)(n-m)(n-m-1)$$

$$+ 2m(n-2)(n-3)\ldots(n-m+1)(n-m)$$

$$+ m(m-1)(n-2)(n-3)\ldots(n-m+1)$$

$= (n - 2)(n - 3)\ldots(n - m + 1)[(n-m)(n-m-1) + 2m(n-m) + m(m-1)]$

$= (n - 2)(n - 3)\ldots(n - m + 1)[(n^2-2nm+m^2-n+m) + 2mn-2m^2+m^2-m]$

$= (n - 2)(n - 3)\ldots( n - m + 1)(n^2 - n)$

$= n(n - 1)( n - 2)(n - 3)\ldots(n - m + 1)$

$= P(n,m).$

---

14-4. Combinations.

The word "combinations" is used in the title of this section and, indeed, of this chapter, in an archaic sense. The use of set-terminology has pushed the word out of the mathematical vocabulary for we cannot speak of a set as a combination of its elements. One may form sets by "collecting" elements; certainly not by "combining" them in any sense of the English word "combine". Atoms may combine to form molecules, but there is a big conceptual difference between a molecule and the set of its atoms.

We never actually use the word in the text, though we do mention it three times.

The method we present for calculating $C(n,m)$ is a standard one, appearing in many algebra texts (among others, Chrystal's Algebra, Volume II (Second Edition, 1900, page 7). Another method appears in Whitworth's Choice and Chance (Fourth Edition, 1886, page 66) and, in a different form, in Kemeny, Snell, and Thompson's Introduction to Finite Mathematics (1957) page 98. The two versions of the second method appear in Section 14-6 of our text.

Exercises 14-4. Answers

1. (a) $2^4 = 16$

   (b) {a,b,c}, {a,b,d}, {a,c,d}, {b,c,d}

   (c) {a,b,c}; abc, acb, bac, bca, cab, cba.

   {a,b,d}; abd, adb, bad, bda, dab, dba.

   {a,c,d}; acd, adc, cad, cda, dac, dca.

   {b,c,d}; bcd, bdc, cbd, cdb, dbc, dcb.

   (d) $C(4,3) = 4$.

2. (a) 45

   (b) 56

   (c) 792

   (d) $C(25,24) = C(25,1) = 25$

   (e) $C(12,10) = C(12,2) = 66$

   (f) $C(100,98) = C(100,2) = 4950$

   (g) $\frac{28}{3}$

   (h) $\frac{C(26,21)}{P(26,5)} = \frac{P(26,5)}{5!} \cdot \frac{1}{P(26,5)} = \frac{1}{120}$

3. $\log 100! = \log(99!100) = \log 100 + \log 99! = 157.9700$.

4. $C(10,8) = C(10,2) = 45$

5. $C(10,2) = 45$

6. $C(15,2) = 105$

7. $C(8,3) = 56$

8. $C(16,4) = 1820$

9. $C(9,4) = 126$, (Void committee not considered).

10. $C(7,2) \cdot C(6,2) = 315$

11. $C(8,2) \cdot C(5,2) = 280$

12. $C(4,1)\cdot C(5,2)\cdot C(6,2) = 600$ if boys team;
$C(4,2)\cdot C(5,2)\cdot C(6,2) = 900$ if girls team.

13. $C(20,2)\cdot C(4,3) = 760$

14. $C(3,2)\cdot C(6,3)\cdot 5! = 7200$

15. $C(10,m-1)$

16. $C(100,5) \approx 7.528 \times 10^7$

17. $C(52,5) \approx 2.599 \times 10^6$

18. $C(100,10) \approx 1.731 \times 10^{13}$

19. $C(4,3)\cdot C(48,2) \approx 4.512 \times 10^3$

20. $P(4,3)\cdot C(13,6)\cdot C(13,6)\cdot C(26,1) \approx 4.596 \times 10^8$

21. $4\cdot C(13,7)\cdot [C(13,2)]^3 \approx 3.258 \times 10^9$

22. $4!C(13,5)\cdot C(13,4)\cdot C(13,3)\cdot C(13,1) \approx 8.209 \times 10^{10}$

23. $C(n,m) = C(n,n-m)$ so $12 = n - 8$ and
$n = 20$. $C(20,17) = C(20,3) = 1140$

24. $4 = 18 - (m + 2)$, $m = 12$. $C(12,5) = 792$.

25. $C(n-1,m-1)+C(n-1,m) = \dfrac{(n-1)!}{(m-1)![(n-1)-(m-1)]!} + \dfrac{(n-1)!}{m![(n-1)-m]!}$

$$= \frac{(n-1)!\,m}{m[(m-1)!][(n-m)!]} + \frac{(n-1)!\,(n-m)}{m![(n-m-1)!](n-m)}$$

$$= \frac{n(n-1)!}{m!(n-m)!}$$

$$= \frac{n!}{m!\,(n-m)!}$$

$$= C(n,m).$$

Alternate proof: The $C(n,m)$ selections can be classified into two sets; those which contain a specified item called "A" and those which do not contain A. There are $C(n-1,m-1)$ selections which contain A and there are $C(n-1,m)$ selections that do not.

$\therefore$ $C(n,m) = C(n-1,m-1) + C(n-1,m)$.

26.

| n \ m | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 1 | 4 | 6 | 4 | 1 | | | | | | |
| 5 | 1 | 5 | 10 | 10 | 5 | 1 | | | | | |
| 6 | 1 | 6 | 15 | 20 | 15 | 6 | 1 | | | | |
| 7 | 1 | 7 | 21 | 35 | 35 | 21 | 7 | 1 | | | |
| 8 | 1 | 8 | 28 | 56 | 70 | 56 | 28 | 8 | 1 | | |
| 9 | 1 | 9 | 36 | 84 | 126 | 126 | 84 | 36 | 9 | 1 | |
| 10 | 1 | 10 | 45 | 120 | 210 | 252 | 210 | 120 | 45 | 10 | 1 |

These values may be found in any table of the binomial coefficients, such as the C.R.C. Standard Mathematical Tables, Twelfth Edition, page 388.

27. $C(n-1,n-2) = \frac{(n-1)!}{(n-2)!} = n-1$, $C(n-2,n-3) = \frac{(n-2)!}{(n-3)!} = n-2$, etc.

Since $C(n,n-2) = \frac{n!}{(n-2)!2!} = \frac{n(n-1)}{2}$ and

$1 + 2 + \ldots + (n-2) + (n-1) = \frac{(n-1)[(n-1)+1]}{2}$

then, $C(n,n-2) = C(n-1,n-2) + C(n-2,n-3) + \ldots + C(2,1) + C(1,0)$ for $3 \leq n$.

---

## 14-5. The Binomial Theorem.

### Exercises 14-5. Answers

1. (a) $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$.

(b) $x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5$

(c) $a^5 + 5a^4b + 10a^3y^2 + 10a^2y^3 + 5ab^4 + b^5$

(d) $a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^4b^3 + 7ab^6 + b^7$

(e) $64u^6 + 192u^5v + 240u^4v^2 + 160u^3v^3 + 60u^2v^4 + 12uv^5 + v^6$

(f) $r^8 - 16r^7s + 112r^6s^2 - 448r^5s^3 + 1120r^4s^4 - 1792r^3s^5 + 1792r^2s^6 - 1024rs^7 + 256s^8$

(g) $\frac{1}{64} + \frac{3}{16}z + \frac{15}{16}z^2 + \frac{5}{2}z^3 + \frac{15}{4}z^4 + 3z^5 + z^6$

(h) $x^9 - \frac{9}{2}x^8 + 9x^7 - \frac{21}{2}x^6 + \frac{63}{8}x^5 - \frac{63}{16}x^4 + \frac{21}{16}x^3 - \frac{9}{32}x^2 + \frac{9}{256}x - \frac{1}{512}$

(i) $x^{16} + 8x^{15} + 28x^{14} + 56x^{13} + 70x^{12} + 56x^{11} + 28x^{10} + 8x^9 + x^8$, or $x^8(x^8 + 8x^7 + 28x^6 + 56x^5 + 70x^4 + 56x^{11} + 28x^2 + 8x + 1)$

(j) $c^{18} - 18c^{17}d + 144c^{16}d^2 - 672c^{15}d^3 + 2016c^{14}d^4 - 4032c^{13}d^5 + 5376c^{12}d^6 - 4608c^{11}d^7 + 2304c^{10}d^8 - 512c^9d^9$, or $c^9(c^9 - 18c^8d + 144c^7d^2 - 672c^6d^3 + 2016c^5d^4 - 4032c^4d^5 + 5376c^3d^6 - 4608c^2d^7 + 2304cd^8 - 512d^9)$

(k) $x^{-6} + 12x^{-5}y^{-2} + 60x^{-4}y^{-4} + 160x^{-3}y^{-6} + 240x^{-2}y^{-8} + 192x^{-1}y^{-10} + 64y^{-12}$

(ℓ) $32x^{-10} - 240x^{-8}y^{-3} + 720x^{-6}y^{-6} - 1080x^{-4}y^{-9} + 810x^{-2}y^{-12} - 243y^{-15}$

2. (a) n

(b) 74; (n + 1)

(c) $17^{th}$

(d) Odd number values

(e) C(35,20)

(f) $42^{\underline{nd}}$ and $31^{\underline{st}}$

(g) 22; C(n,5) = C(n,17) and $5 = n = 17$

(h) $C(22,11)\ a^{11}b^{11} = 705,432\ a^{11}b^{11}$

3. (a) $C(15,6)a^9b^6 = 5005a^9b^6$

(b) $C(13,3)x^{10}(-5)^3 = -35,750x^{10}$

(c) $C(13,11)(2x)^2(-1)^{11} = -78x^2$

(d) $C(10,5)(3x^{-1})^5(\frac{x}{3})^5 = 252$

(e) $C(12,6)(x^{-1})^6(x^2)^6 = 924x^6$

(f) $C(14,7)(1)^7(-\frac{x^2}{2})^7 = \frac{429}{16}x^{14}$

(g) $C(10,7)a^3b^7 = 120a^3b^7$

(h) $C(9,4)(x^2)^4(-y)^5 = -126x^8y^5$

(i) $C(10,8)(\frac{2}{x})^2(-x^2)^8 = 180x^{14}$

(j) $C(9,6)(x^3)(-2y\frac{1}{2})^6 = 5376x^3y^6$

(k) $C(12,8)(x^2)^8(\frac{1}{x})^8 = 495$

4. (a) $1.02^4 = 1 + 4(0.02) + 6(0.02)^2 + 4(0.02)^3 + (0.02)^4$

$= 1 + 0.08 + 0.0024 + 0.000032 + 0.00000016$

$\approx 1.0824$

(b) $1.02^{12} = 1 + 12(0.02) + 66(0.02)^2 + 220(0.02)^3 + 495(0.02)^4 + \ldots$

$= 1 + 0.24 + 0.0264 + 0.00176 + 0.00007920 + \ldots$

$\approx 1.2682$

(c) $(1 - 0.02)^{12} = 1 - 0.24 + 0.0264 - 0.00176 + 0.00007920 - \ldots$

$\approx 0.7847$

(d) $2.01^{10} = 2^{10} + 10(2)^9(0.01) + 45(2)^8(0.01)^2 + 120(2)^7(0.01)^3 + 210(2)^6(0.01)^4 + \ldots$

$= 1024 + 51.20 + 1.1520 + 0.015360 + 0.00001920 + \ldots$

$\approx 1076.3674$

(e) $(2-0.01) = 1024 - 51.20 + 1.1520 - 0.015360 + 0.00001920 - \ldots$

$\approx 973.9367$

(f) $16; (1-1)^8 = [(1-1)^4]^2 = (1-4+6-4+1^4)^2 = (-4)^2 = 16$

(g) $(2-1)^5 = 2^5 - 5(2)^4 1 + 10(2)^3(1)^2 - 10(2)^2(1)^3 + 5(2)(1)^4 - 1^5 = 32 - 31$

(h) $(\frac{1}{2} + \frac{\sqrt{3}}{2}1)^7 = (\frac{1}{2})^7 + 7(\frac{1}{2})^6(\frac{\sqrt{3}}{2}1) + 21(\frac{1}{2})^5(\frac{\sqrt{3}}{2}1)^2 + 35(\frac{1}{2})^4(\frac{\sqrt{3}}{2}1)^3 + 35(\frac{1}{2})^3(\frac{\sqrt{3}}{2}1)^4 + 21(\frac{1}{2})^2(\frac{\sqrt{3}}{2}1)^5 + 7(\frac{1}{2})(\frac{\sqrt{3}}{2}1)^6 + (\frac{\sqrt{3}}{2}1)^7$

$= \frac{1}{128} + \frac{7\sqrt{3}}{128}1 - \frac{63}{128} - \frac{105\sqrt{3}}{128}1 + \frac{315}{128} + \frac{139\sqrt{3}}{128}1 - \frac{189}{128} - \frac{27\sqrt{3}}{128}1 =$

$= \frac{1}{2} + \frac{\sqrt{3}}{2}i$

Note: Students may remember from Chapter 12 that $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$ is a cube root of unity, so that

$$\left(\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i\right)^7 = \left[-\left(-\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i\right)^3\right]^2 \left[-\left(-\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i\right)\right]$$

$$= 1 \cdot \left(\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i\right)$$

---

14-6. Arrangements.

In this section we consider the number of ways a list having duplications may be (re-) arranged. Using the same reasoning we determine the number of ordered partitions of a finite set. Compare Kemeny, Snell, and Thompson, An Introduction to Finite Mathematics. As a corollary we obtain another method for counting the m-element subsets of an n-element set.

Either or both of Sections 14-6, 14-7 may be omitted at the discretion of the teacher.

Exercises 14-6a; Answers

1. $\frac{6!}{1!3!2!} = 120$

2. $\frac{5!}{2!3!} = 10$

3. $\frac{14!}{3!3!2!2!2!} = 201,\ 801,\ 600$

4. $\frac{52!}{(24)^6} \approx 4.114 \times 10^{33}$

5. $6! = 720$; use "qu" as a single element

6. $\frac{8!}{4!2!} - \frac{5}{2!} = 780$; use "$\ell\ell\ell\ell$" as an element.

7. $\frac{6!}{2!2!2!} = 90$

8. There are 3!, 3-letter arrangements having no 0; $C(3,2)\cdot 3!$ arrangements having one 0 and $C(3,1)\cdot 3$ arrangements having two 0's. $3! + 3\cdot 3! + 3\cdot 3 = 33$.

9. $4! + 3\cdot\frac{4!}{2!} = 60$

*Exercises 14-6b; Answers.*

1. $\frac{8!}{(2!)^4} = 2520$

2. $\frac{8!}{2!3!3!} = 560$

3. $\frac{40!}{10!30!} \approx 8.472 \times 10^8$

4. $\frac{8!}{3!5!} = 56$

5. $\frac{52!}{(13!)^4} \approx 5.364 \times 10^{28}$

6. (a) $\frac{6!}{(2!)^3(3!)} = 15$

(b) $\frac{12!}{(3!)^4(4!)} = 15{,}400$

(c) $\frac{(n,k)!}{(k!)^n(n!)}$

---

14-7. Selections with Repetitions.

*This section involves calculations which are rather more involved than those in the earlier sections. Since it is the last one in the chapter, it goes without saying that it may be omitted at the the teacher's discretion without loss of "continuity".*

*We apply our result to a problem (Example 14-7c) of* partitio numerorum. *The general problem in this field is to determine the number of ways (if any) of representing a natural number n as a sum of terms taken from a given set of integers. The number of terms may be either restricted (as in our example) or unrestricted and we may either consider or ignore the order of the terms. The two problems of this sort which we consider (Example 14-4e and Example 14-7c) are two of the easier ones in this field. (In our examples the given sets are the natural numbers and the non-negative integers). This branch of mathematics is extensive and fascinating.*

It abounds with unsolved problems, ingenious methods and striking results. We can go into none of this here. (Cf. Hardy and Wright, An Introduction to the Theory of Numbers, Oxford University Press, Chapters 19, 20, 21.)

One of the most famous of the unsolved problems in this theory - and one of the simplest to state - is the question whether every even natural number greater than 2 is a sum of two primes. Goldbach conjectured (1742) that the answer is "yes", that given any natural number $n$, greater than 1, there are primes $p$ and $q$ (not necessarily distinct) for which

$$2n = p + q$$

No one has proved this nor has anyone found an even natural number which is not the sum of two primes. Thus, we don't even know whether the number of such representations is always positive. Here is an easily stated counting problem which has eluded solutions for over 200 years.

Exercises 14-7; Answers

1. $\binom{10 + 12 - 1}{12} = 293{,}930$

2. $\binom{10 + 5 - 1}{5} = 2002$

3. $\binom{6 + 5 - 1}{5} = 252$

4. $\binom{6 + 2 - 1}{2} = 21$

5. $\binom{13 + 2 - 1}{2} = 91$

6. This problem is analogous to Example 14-7c.

$$\binom{21 + 5 - 1}{5 - 1} = \frac{25!}{4!21!} = 12{,}650.$$

---

14-8, Miscellaneous Exercises. Answers

The first 30 of these exercises are arranged "in order of difficulty". The 31st, although verbose, has a "moral" lesson.

1. $\frac{5!}{2!2!} = 30$

2. $C(10,7) = C(10,3) = 120$

3. $C(12,2) = 66$

4. $\frac{n(n-3)}{2}$; There are $C(n,2)$ lines on the n-points, but n of these are sides of the polygon, so the number of diagonals is $C(n,2) - n = \frac{n!}{2!(n-2)!} - n = \frac{n(n-3)}{2}$.

5. $6! = 720$

6. $2 \cdot P(6,4) = 720$

7. $C(5,3) = 10$

8. $2 \cdot 8! = 80,640$

9. $5 \cdot P(8,3) = 1680$

10. $C(10,3) \cdot C(5,2) \cdot 5! = 144,000$

11. $26 \cdot 25^4 = 10,156,250 \approx 1.016 \times 10^7$.

12. $3 \cdot 2 \cdot 2 \cdot 1 \cdot 1 = 12$.

13. $C(2,2) + C(3,2) = 4$; the even sum arises from a pair of odds or a pair of evens.

14. $5(4)(3) = 60$

15. $C(4,3)\ 12 \cdot C(4,2) = 288$ Following the queen's selection in $C(4,3)$ ways, there are 12 choices for the type of pair and $C(4,2)$ ways to form the pair. Or, $C(4,3)$ ways for queens with 48 choices for the first card of the pair and 3 choices for the second, but $\frac{48 \cdot 3}{2}$ ways to form pair to avoid duplications. Hence,

$C(4,3) \cdot \frac{48 \cdot 3}{2} = 288$.

16. $C(4,3) + C(4,2) = C(5,3) = 10$. At least one flag of each color must be used. Case (i); if 1 red flag and 3 blue, the $C(4,1)$ ways for the red flag to fill one of four positions. Case (ii); if 2 red flags used, then $C(4,2)$ ways.

17. $6(3)(2)(2)(1)(1) = 72$

18. $5!7!3!3! = 21,772,800.$

19. $8^3 = 512.$

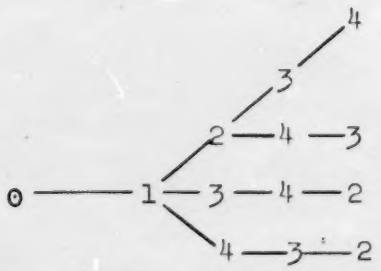
20. $\frac{8!}{2} = 20,160.$ Since the steel ring can be looked at from both sides (or "turned over"), each only half of the 8! arrangements will be distinct.

21. $2 \cdot 5 \cdot P(5,5) + 5 \cdot 4 \cdot P(5,5) = 3600.$ One of the two who will not sit next to each other may select an end seat in 2 ways, leaving 5 places for the others and $P(5,5)$ ways for the remaining 5 persons. There are 5 ways to not sit in an end seat, leaving 4 ways for the second person and $P(5,5)$ ways for the remaining persons.

22. $\sum_{k=3}^{6} P(6,k) = 1920$

23. $C(10,7) - C(8,5) = 64,$ where $C(8,5)$ ways to have the two friends together at the dinner party.

24. 8; let 1,2,3,4 represent the relative heights. Then a tree displays the different way for one order starting with the shortest boy at one end. Since, the shortest boy may also be placed at the opposite end, the number of branches in the tree must be doubled.



25. $P(6,5) = 720$

26. $2^6 - 1 = 63$

27. 1440. Three cases can be considered. Case (i). Vowel in first position, then there are six ways to keep vowels separated. Case (ii). Vowel in second position, then there are three ways to keep vowels separated. Case (iii). Vowel in third place, then there is but one way to keep the vowels separated. Permuting the vowels and consonents in these cases gives the total number of ways as- $6 \cdot 3!4! + 3 \cdot 3!4! + 1 \cdot 3!4! = 10 \cdot 3! \cdot 4! = 1440$.

28. $3(5)(5) = 75$, where there are three choices for the final digit, five choices for the first digit, and five choices for the second digit.

*29. $3(7)(6) = 126$, where there are three choices for the final digit, six choices for the first digit, and seven choices for the second digit.

*30. $4(7)(6) + 4(5) + 2 = 190$; $1(6)(5) + 3(5)^2 + 3(4) + 2 = 119$.

31. (a) $C(mn,2)$

(b) $n \cdot C(m,2)$

(c) $\dfrac{n \cdot C(m,2)}{C(mn,2)} = \dfrac{m - 1}{mn - 1}$

| n \ m | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| 2 | $\frac{1}{3}$ | $\frac{2}{5}$ | $\frac{3}{7}$ | $\frac{4}{9}$ |
| 3 | $\frac{1}{5}$ | $\frac{1}{4}$ | $\frac{3}{11}$ | $\frac{2}{7}$ |
| 4 | $\frac{1}{7}$ | $\frac{2}{11}$ | $\frac{1}{5}$ | $\frac{4}{19}$ |
| 5 | $\frac{1}{9}$ | $\frac{1}{7}$ | $\frac{3}{19}$ | $\frac{1}{6}$ |

(d) (i) Chance of exposure becomes less. (ii) Chance of exposure becomes greater.

## 14-9. Illustrative Test Questions

A. True-False items.

1. $P(n,n) = C(n,n) = 1$
2. $P(n,m) = C(n,m) \cdot m!$
3. $C(n,n) = P(n,n)$
4. $P(5,3) = C(5,2)$
5. $C(8,3) = C(7,3) + C(7,2)$
6. The number of three-element subsets which can be selected from a set of five elements is $3!$
7. The number of pairs of elements which can be selected from a set of eight elements is $\frac{8!}{2!6!}$.
8. $(n + 1)! = (n + 1)n!,\ n \neq 0$
9. $(n!)(m!) = (n + m)!,\ n,m \neq 0$
10. $\frac{n!}{(n - 2)!} = (n - 1)!,\ 2 < n$

B. Multiple choice items.

1. How many four digit numbers may be formed from the digits 1,2,3,4,5?

   A. $C(5,4)$ D. $5^4$

   B. $P(5,4)$ E. None of the above is correct.

   C. $4^5$

2. If a nickel and two pennies are laid in a row, in how many ways can they show heads or tails?

   A. 16 D. 4

   B. 8 E. 3

   C. 5

3. If two distinguishable dice are tossed, in how many ways can they show a total of seven?

A. 3 D. 6

B. 4 E. 7

C. 5

4. How many ways can three different paintings be arranged on the four walls of a room, one painting to a wall?

A. $3^4$ D. $3!$

B. $4^3$ E. $3 \times 4$

C. $4!$

5. How many ways can the letters of the word LINK be arranged if none of the resulting arrangements may begin with K?

A. 81 D. 18

B. 27 E. 6

C. 24

6. If there are 10 teams in a basketball league, how many games would have to be played if each team plays every other team exactly once?

A. 5 D. 45

B. 20 E. 90

C. 25

7. How many different outfits consisting of a sweater, a skirt, and a pair of shoes can a girl wear if she has 3 sweaters, 4 skirts, and 2 pairs of shoes?

A. 48 D. 12

B. 36 E. None of the above is correct.

C. 24

8. From a group of 7 boys and 8 girls, how many ways can a committee of 3 boys or 3 girls be chosen?

A. $C(7,3) \cdot C(8,3)$

B. $C(7,3) + C(8,3)$

C. $C(15,3)$

D. $(8 \cdot 7 \cdot 6) + (7 \cdot 6 \cdot 5)$

E. $C(15,6)$

9. How many sums can be formed from a penny, a nickel, a dime, a quarter, and/or a half dollar?

A. $5!$

B. $2^5$

C. $2^5 - 1$

D. $5^2$

E. None of the above is correct.

10. Which of the following expressions gives the number of three letter symbols that can be formed from the letters of the word MATHEMATICS if no letter is repeated in a symbol?

A. $\frac{11!}{8!}$

B. $\frac{8!}{5!}$

C. $\frac{8!}{3!}$

D. $\frac{11!}{3!}$

E. $\frac{11!}{5!}$

11. The number of distinguishable ways the letters of the name TENNESSEE can be arranged is:

A. $9!$

B. $\frac{9!}{3!}$

C. $2^9$

D. $\frac{9!}{(4!)(2!)(2!)}$

E. $\frac{9!}{2!}$

12. If n coins are tossed, how many ways may they fall?

A. $n^2$ D. $n!$

B. $2n$ E. $2n!$

C. $2^n$

13. The expression $15 \cdot 14 \cdot 13 \cdot 12$ is equal to:

A. $\frac{15!}{12!}$ D. $\frac{(15!)(13)(12)}{11!}$

B. $\frac{15(14!)}{11!}$ E. None of the above is correct.

C. $\frac{15!}{(4!)(11!)}$

14. The value of $\frac{(6!)(0!)}{3!}$ is:

A. 0 D. The expression cannot be evaluated.

B. Undefined E. None of the above is correct.

C. 20

15. $(n - r - 2)!\ (n - r - 1)(n - r)$ is equal to

A. $(n - r)!$ D. $\frac{n!}{(n - r)!}$

B. $\frac{(n - r)!}{(n - r - 2)!}$ E. None of the above is correct.

C. $\frac{n!}{r!}$

16. If $P(n,3) = 120$, then n is equal to:

A. 12 D. 5

B. 10 E. 4

C. 6

17. If $C(n,r) = 10$ and $P(n,r) = 60$, then $n$ is equal to:

    A. 3 D. 10

    B. 5 E. 50

    C. 6

18. In the expansion of $(a + x)^n$, where $n$ is a positive integer,

    A. the first term is $a^n x$

    B. the number of terms is $n$

    C. the sum of the exponents of $a$ and $x$ in each term is $n + 1$

    D. the next to the last term is $nax^{n-1}$

    E. All the above are correct.

C. Short answer items.

1. Find the value of $P(10,3)$

2. Find the value of $C(80,78)$

3. Find the value of $\frac{12!}{(4!)(6!)}$

4. Find $n$ if $\frac{n!}{(n-2)!} = 56$

5. How many odd numbers having three digits can be formed from the digits 0, 1, 2, 3, 4, 5?

6. How many ways can 2 girls and 4 boys be seated in a row of 6 chairs if the 2 girls sit side by side?

7. If there are 4 plane routes and 5 rail routes between Chicago and St. Paul, how many ways can a man complete a round trip if he always travels one way by plane and one way be rail?

8. How many code words each containing four letters can be composed from 3 vowels and 4 consonants if the vowels and consonants must alternate?

9. If $C(n,6) = C(n,8)$, find $C(n,2)$

10. Find the <u>two</u> terms in the expansion of $(a + b)^6$ whose coefficients are equal to $C(6,2)$.

11. Find the fifth term in the expansion of $(x + y)^7$

D. <u>Problems</u>

1. How many ways may three glasses be filled without mixing, if there are five kinds of wine available?

2. How many different "words" may be formed as arrangements of the letters of <u>COCOA</u> so the letter A lies in the center of the arrangement?

3. How many different arrangements of the letters of <u>SYZYGY</u> may be made so the three y's do <u>not</u> come together?

4. How many different arrangements of the letters of <u>UBIQUITOUS</u> may be formed so the letter q is followed by the letter u?

5. How many committees consisting of two or more persons may be formed from a group of ten persons?

6. How many lines are determined by a set of nine points on a plane if one of the lines lies on four of the points and no one of the other lines lies on more than two of the points?

7. Six packages are to be delivered to six different addresses. If two delivery boys are available, how many different ways may they be delivered?

## 14-9 Illustrative Test Items, Answers

A. True-false items.

1. False
2. True
3. False
4. False
5. True
6. False
7. True
8. True
9. False
10. False

B. Multiple choice items

1. D
2. B
3. D
4. C
5. D
6. D
7. C
8. B
9. C
10. B
11. D
12. C
13. B
14. E
15. A
16. C
17. B
18. D

C. Short answer items.

1. 720
2. 3160
3. 27,720
4. 8
5. 90
6. 240
7. 40
8. 288
9. 91
10. $15a^2b^4$ and $15a^4b^2$
11. $35x^2y^5$

D. Problems

1. $\binom{5+3-1}{3} = 35$

2. $\frac{4!}{2!2!} = 6$

3. $\frac{6!}{3!} - 4! = 96$

4. $\frac{9!}{2!2!} = 90{,}720$

5. $2^{10} - 1 - 10 = 1013$

6. $C(9,2) - C(4,2) + 1 = 31$

7. $2^6 = 64$, since each package has two ways in which it may be delivered.

**Exercises 15-2; Answers.**

1.

| | a | b |
|---|---|---|
| a | a | b |
| b | a | a |

,

| | a | b |
|---|---|---|
| a | a | a |
| b | b | a |

,

| | a | b |
|---|---|---|
| a | b | a |
| b | a | a |

,

| | a | b |
|---|---|---|
| a | a | a |
| b | b | b |

,

| | a | b |
|---|---|---|
| a | a | b |
| b | a | b |

,

| | a | b |
|---|---|---|
| a | b | b |
| b | a | a |

,

| | a | b |
|---|---|---|
| a | b | a |
| b | b | a |

,

| | a | b |
|---|---|---|
| a | b | a |
| b | a | b |

,

| | a | b |
|---|---|---|
| a | a | b |
| b | b | b |

,

| | a | b |
|---|---|---|
| a | b | a |
| b | b | b |

,

| | a | b |
|---|---|---|
| a | b | b |
| b | a | b |

,

| | a | b |
|---|---|---|
| a | b | b |
| b | b | a |

.

2. That is an operation in A follows from the fact that the product in the conventional sense of members a and b of A is itself a member of A. The multiplication table is:

| • | 1 | i | -1 | -i |
|---|---|---|---|---|
| 1 | 1 | i | -1 | -i |
| i | i | -1 | -i | 1 |
| -1 | -1 | -i | 1 | i |
| -i | -i | 1 | i | -1 |

**Exercises 15-3; Answers.**

.. Here only Example 4 calls for comment. Suppose that $\alpha$ and $\beta$ are nth roots of 1. From $\alpha^n = 1$ and $\beta^n = 1$, we

have $(\alpha \beta)^n = 1$ and $(\alpha/\beta)^n = 1$. That is, $\alpha\beta$ and $\alpha/\beta$ are each nth roots of 1. From the fact that $\alpha\beta$ is an nth root of 1, we see that $\cdot$ is an operation in A. From the fact $\alpha/\beta$ is an nth root of 1, we see that Postulate G 2 is fulfilled, the uniqueness of solution of the equation $\beta z = \alpha$ in A being guaranteed by the uniqueness of the solution of $\beta z = \alpha$ in C. The associative law follows automatically from the fact that multiplication in the complex number system is associative. Note that $\cdot$ is commutative. Consequently the equation $z\beta = \alpha$ has exactly the same solution set in A as does $\beta z = \alpha$.

2. Not every equation of the form $\alpha z = \beta$ where $\alpha$ and $\beta$ are given complex numbers has a solution; e.g., take $\alpha = 0$, $\beta = 1$.

3. Given a, b, c, d integers, we have

$$(a + b\sqrt{2}) + (c + d\sqrt{2}) = (a + c) + (b + d)\sqrt{2} \in A,$$

since $a + c$ and $b + d$ are integers. Also the equation

$$(a + b\sqrt{2}) + x = c + d\sqrt{2}$$

has the unique solution

$$(c - a) + (d - b)\sqrt{2}$$

in R, and moreover this solution is a member of A since $c - a$ and $d - b$ are both integers. The remaining details are readily furnished.

4. See Section 7 of this chapter "Subfields intermediate to Q and R".

---

Exercises 15-4; Answers.

1. Example 2: the inverse of a is $1/a$.

   Example 3: the inverse of (a,b,c) is (-a, -b, -c).

Example 4: the inverse of $\alpha = \cos(\frac{2\pi k}{n}) + i \sin(\frac{2\pi k}{n})$, $k = 0, 1, \ldots, n-1$, is $1/\alpha = \cos(\frac{2\pi k}{n}) - i \sin(\frac{2\pi k}{n})$.

Example 5: the inverse of $a$ is $1/a$.

Exercise 3: the inverse of $a + b\sqrt{2}$ is $(-a) + (-b)\sqrt{2}$.

Exercise 4: the inverse of $a + b\sqrt{2}$ is

$$\left(\frac{a}{a^2 - 2b^2}\right) + \left(\frac{-b}{a^2 - 2b^2}\right)\sqrt{2}.$$

2.

$$\begin{aligned} a \cdot (a^{-1} \cdot b) &= (a \cdot a^{-1}) \cdot b \\ &= e \cdot b \\ &= b. \end{aligned}$$

$$\begin{aligned} (b \cdot a^{-1}) \cdot a &= b \cdot (a^{-1} \cdot a) \\ &= b \cdot e \\ &= b. \end{aligned}$$

3. The table for $\{0,1\}$ does not satisfy the group requirements. The equation $0 \cdot x = 1$ does not have a solution in $\{0,1\}$. The first three tables given for $\{a,b\}$ do not satisfy the group requirements, for in the cases of the first and third tables the equation $a \cdot x = b$ has no solution in $A$ and in the case of the second table the equation $b \cdot x = a$ has no solution in $A$.

The fourth table for $\{a,b\}$ does satisfy the group requirements. That G 2 is satisfied may be seen by noting that each new row and each column of the body of the table contain each of the elements $a$ and $b$ (without repetition).

Notice that we cannot be cavalier about the associative law! We must examine the 8 cases afforded by the distinct ordered triples with components in A. The confirmation of the associative law is given by the following table.

| $c_1$ | $c_2$ | $c_3$ | $c_1 \cdot (c_2c_3)$ | $(c_1c_2) \cdot c_3$ |
|---|---|---|---|---|
| a | a | a | $a \cdot (aa) = a \cdot a = a$ | $(aa) \cdot a = a \cdot a = a$ |
| a | a | b | $a \cdot (ab) = ab = b$ | $(aa) \cdot b = a \cdot b = b$ |
| a | b | a | $a(ba) = ab = b$ | $(ab)a = ba = b$ |
| a | b | b | $a(bb) = aa = a$ | $(ab)b = bb = a$ |
| b | a | a | $b(aa) = ba = b$ | $(ba)a = ba = b$ |
| b | a | b | $b(ab) = bb = a$ | $(ba)b = bb = a$ |
| b | b | a | $b(ba) = bb = a$ | $(bb)a = aa = a$ |
| b | b | b | $b(bb) = ba = b$ | $(bb)b = ab = b$ |

Each of the indicated reductions in the second and third columns of the body of the table is carried out by use of the multiplication table with which we are concerned. We have: $e = a$, $a^{-1} = a$, $b^{-1} = b$.

The table

| $\cdot$ | a | b |
|---|---|---|
| a | b | b |
| b | a | a |

yields an example of a <u>non</u>-associative operation. In fact, $(aa)b = bb = a$ and $a(ab) = ab = b$, so that $(aa)b \neq a(ab)$, a being distinct from b.

4. Suppose that e and f are elements of A satisfying for each $a \in A$.

$$ae = ea = a,$$

$$af = fa = a.$$

Then setting $a = f$ in the first line, we obtain

$$fe = f,$$

and setting $a = e$ in the second line, we obtain

$$fe = e.$$

Hence

$$e = f.$$

It follows that there is at most one element $e \in A$ satisfying for all $a \in A$: $ae = ea = a$.

5. We have

$$a(xb) = (ax)b = eb = b\ ,$$

so that $xb$ is a solution of $az = b$. Thus $az = b$ has at least one solution. If $z$ is any solution of $az = b$, we have

$$yb = y(az) = (ya)z = ez = z,$$

so the only possibility for $z$ is the element $yb$. Thus $az = b$ has at most one solution in $A$. Hence the equation $az = b$ has a unique solution in $A$.

The equation $wa = b$ is similarly treated.

Corollary. $x = y$.

We found (i) $xb$ satisfies $az = b$, (ii) no member of $A$ besides $yb$ satisfies $az = b$. It follows that $xb = yb$. But $b$ is arbitrary. Taking $b = e$, we obtain $x = y$.

(Thus a "right" inverse is also a "left" inverse -- even if our operation is non-commutative, provided each of them exists. We neither knew nor needed this fact in solving Exercise 15-4, 5, however.)

6. Since e is the identity element, the following part of the table is evident:

| . | e | a | b |
|---|---|---|---|
| e | e | a | b |
| a | a | | |
| b | b | | |

Consider the product aa. It is not possible that aa = a, for ae = a and the equation ax = a has a unique solution.

It is not possible that aa = e, for if aa = e, then

$$ab = b$$

since the equation ax = b has a solution in A and this solution would have to be distinct from e and a. Since

$$eb = b,$$

and the equation yb = b has a unique solution, we should be forced to conclude that a = e. This is impossible. We must reject aa = e. Hence necessarily aa = b.

At this stage we are assured that our table contains the following entries:

| . | e | a | b |
|---|---|---|---|
| e | e | a | b |
| a | a | b | |
| b | b | | |

Since the element a has an inverse of $a^{-1}$ and neither e nor a is the inverse of a (as we see from the second line of the table as far as it has been constructed), $a^{-1} = b$.

Hence $ab = ba = e$. We have at this stage

| $\cdot$ | e | a | b |
|---|---|---|---|
| e | e | a | b |
| a | a | b | e |
| b | b | e | |

We now see, since the equation $bx = a$ has a solution in $A$ and this solution is different from $e$ and $a$, that $bb = a$. Conclusion: if we have a group containing precisely three elements: e, a, b, and e is the identity element, the multiplication table is

(*)

| $\cdot$ | e | a | b |
|---|---|---|---|
| e | e | a | b |
| a | a | b | e |
| b | b | e | a |

We must note that we have merely shown that, if $(A, \cdot)$ is a group, then the multiplication table is given by (*). There remains to be shown that (*) does respect the group axioms.

G 2. Since each row and column of the body of (*) contains each of the elements of $A$ precisely once, G 2 is satisfied.

G 1. We may break down the checking of the associative law into two cases.

Case 1. At least one of the factors is e. This case is disposed of by noting

$$(ec_2)c_3 = c_2c_3 = e(c_2c_3), \qquad c_2, c_3 \in A;$$

$$(c_1e)c_3 = c_1c_3 = c_1(ec_3), \qquad c_1, c_3 \in A;$$

$$(c_1c_2)e = c_1c_2 = c_1(c_2e), \qquad c_1, c_2 \in A.$$

Case 2. No factor is e. We list all the possibilities and compute the desired products employing (*).

| $c_1$ | $c_2$ | $c_3$ | $(c_1c_2)\cdot c_3$ | $c_1\cdot(c_2c_3)$ |
|---|---|---|---|---|
| a | a | a | (aa)a = ba = e | a(aa) = ab = e |
| a | a | b | (aa)b = bb = a | a(ab) = ae = a |
| a | b | a | (ab)a = ea = a | a(ba) = ae = a |
| a | b | b | (ab)b = eb = b | a(bb) = aa = b |
| b | a | a | (ba)a = ea = a | b(aa) = bb = a |
| b | a | b | (ba)b = eb = b | b(ab) = be = b |
| b | b | a | (bb)a = aa = b | b(ba) = be = b |
| b | b | b | (bb)b = ab = e | b(bb) = ba = e |

Exercises 15-5; Answers.

1. Here $n \circ \ell(x) = \lambda(\alpha x + \beta) + \mu$. From $n \circ \ell = m$, we conclude that $\lambda\alpha = \gamma$ and $\lambda\beta + \mu = \delta$. Hence $\lambda = \gamma/\alpha$, $\mu = \delta - (\beta\gamma/\alpha)$. With $\lambda$ and $\mu$ so taken $n \circ \ell = m$.

2. The identity element is the linear function e given by $e(x) = 1 \cdot x + 0 = x$.

3. From $\ell \circ n = e$, we have $\lambda = 1/\alpha$, $\mu = -\beta/\alpha$.

4. $\ell(x) = \alpha x + \beta$, $m(x) = \gamma x + \delta$, $\ell^{-1}(x) = \frac{1}{\alpha}x + (\frac{-\beta}{\alpha})$.

$\ell^{-1} \circ m(x) = \frac{1}{\alpha}(\gamma x + \delta) + (\frac{-\beta}{\alpha}) = (\frac{\gamma}{\alpha})x + \frac{\delta - \beta}{\alpha}$.

$\ell \circ (\ell^{-1} \circ m)(x) = \alpha[(\frac{\gamma}{\alpha})x + \frac{\delta-\beta}{\alpha}] + \beta = \gamma x + \delta$.

$m \circ \ell^{-1}(x) = \gamma[\frac{1}{\alpha}x + (\frac{-\beta}{\alpha})] + \delta = \frac{\gamma}{\alpha}x + \frac{\alpha\delta - \beta\gamma}{\alpha}$.

$(m \circ \ell^{-1}) \circ \ell(x) = \frac{\gamma}{\alpha}(\alpha x + \beta) + \frac{\alpha\delta - \beta\gamma}{\alpha} = \gamma x + \delta$.

5. We have $\ell \circ m(x) = \alpha\, \gamma\, x + (\beta + \alpha\, \delta)$ and $m \circ \ell\,(x) = \gamma\, \alpha\, x + (\delta + \gamma\, \beta)$. Hence $\ell \circ m = m \circ \ell$ if and only if $\beta + \alpha\, \delta = \delta + \gamma\, \beta$. This latter equality holds if and only if $\alpha\, \delta - \delta = \gamma\, \beta - \beta$. The assertion follows.

6. Note that, if $(a,b)$, $(c,d) \in A$, then $(a,b) \cdot (c,d) = (ac,\ ad + b) \in A$ since $ac \neq 0$. Given elements $(a_1, b_1)$, $(a_2, b_2)$, $(a_3, b_3) \in A$, we have

$$((a_1, b_1) \cdot (a_2, b_2)) \cdot (a_3, b_3) = (a_1a_2,\ a_1b_2 + b_1) \cdot (a_3, b_3)$$
$$= (a_1a_2a_3,\ a_1a_2b_3 + (a_1b_2 + b_1)),$$

and

$$(a_1, b_1) \cdot ((a_2, b_2) \cdot (a_3, b_3)) = (a_1, b_1) \cdot (a_2a_3,\ a_2b_3 + b_2)$$
$$= (a_1a_2a_3,\ a_1\,(a_2b_3 + b_2) + b_1).$$

The associative law now follows.

Note that for every $(a,b) \in A$, we have

$$(a,b) \cdot (1,0) = (1,0) \cdot (a,b) = (a,b).$$

Hence $A$ has an identity element, namely $(1,0)$. Further, $(\frac{1}{a}, -\frac{b}{a})$ satisfies both

$$(a,b) \cdot (x,y) = (1,0)$$

and

$$(x,y) \cdot (a,b) = (1,0).$$

The conditions of Ex. 15-4, No. 5 are fulfilled. $(\frac{1}{a}, -\frac{b}{a})$ is the inverse of $(a,b)$.

A (1,1) correspondence between A and the set of non-constant linear functions is defined by the rule which assigns to $(a,b) \in A$ the linear function given by

$$\ell(x) = ax + b.$$

This correspondence has the property that if m corresponds to $(c,d) \in A$, then $m \circ \ell$ corresponds to $(c,d) \cdot (a,b)$. That is, "product corresponds to product". This is an instance of isomorphism. The structure (A, ) was, of course, constructed in an obvious way from the group of non-constant linear functions with composition as the operation. The object of the exercise was to construct a group isomorphic to an important group of common occurrence but having elements and rules of a different nature.

7. This exercise is straightforward. It suffices to note in either case that $\cdot$ is an operation, that $(1,0) \in A$ is the identity element, that, if $(a,b) \in A$, then $(\frac{1}{a}, -\frac{b}{a}) \in A$ and that the verification of the associative law remains valid for the case where A consists of the set of ordered pairs of complex numbers with non-zero first components.

---

Exercises 15-6; Answers.

1. We note that $(bd)(b^{-1}d^{-1}) = 1$, so that $(bd)^{-1} = b^{-1}d^{-1}$. Hence

$$\begin{aligned} \frac{ad + bc}{bd} &= (bd)^{-1}(ad + bc) \\ &= b^{-1}d^{-1}(ad + bc) \\ &= (b^{-1}d^{-1})(ad) + (b^{-1}d^{-1})(bc) \\ &= b^{-1}a + d^{-1}c \\ &= \frac{a}{b} + \frac{c}{d}. \end{aligned}$$

The deta[illegible] are readily supplied.

2. The argument may be based on the use of reciprocals. Thus

$$(a/b)/c = c^{-1} \cdot (b^{-1}a)$$
$$= (b^{-1}c^{-1}) \cdot a$$
$$= (bc)^{-1}a$$
$$= a/bc.$$

The second part may be treated as follows.

$$(a/b)/(c/d) = (b^{-1}a)/(d^{-1}c)$$
$$= (d^{-1}c)^{-1}\,(b^{-1}a)$$
$$= ((d^{-1})^{-1}c^{-1})(b^{-1}a)$$
$$= (bc)^{-1}(ad)$$
$$= ad/bc.$$

The following points should be emphasized:

(a) The indicated calculations in the asserted identity are all meaningful, there being no divisions by zero.

(b) $(d^{-1})^{-1} = d$.

(c) A corresponding result holds for an arbitrary abelian group.

3. The given pair of equations imply

$$\begin{cases} e(ax + by) = ce \\ b(dx + ey) = bf \end{cases} \qquad \begin{cases} d(ax + by) = cd \\ a(dx + ey) = af \end{cases}$$

and subtraction gives (respectively)

$$(ae - bd)x = ce - bf, \qquad (ae - bd)y = af - cd.$$

Since $ae - bd \neq 0$, we conclude

$$x = \frac{ce - bf}{ae - bd}, \qquad y = \frac{af - cd}{ae - bd};$$

so that <u>if</u> our system has any solution $(x,y)$ it must be

$$\left(\frac{ce - bf}{ae - bd}, \frac{af - cd}{ae - bd}\right).$$

Substitution in the original equations verifies that this couple is indeed a solution:

$$a \cdot \frac{ce - bf}{ae - bd} + b \cdot \frac{af - cd}{ae - bd} = \frac{ace - abf + abf - bcd}{ae - bd} = c$$

$$d \cdot \frac{ce - bf}{ae - bd} + e \cdot \frac{af - cd}{ae - bd} = \frac{cde - bdf + aef - cde}{ae - bd} = f.$$

4.

| + | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 1 | 2 |
| 1 | 1 | 2 | 0 |
| 2 | 2 | 0 | 1 |

| · | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 |
| 2 | 0 | 2 | 1 |

Both commutative laws follow from the very construction of the addition and multiplication tables. On turning to the table (*) of Ex. 15-4, No. 6, we see on taking $e = 0$, $a = 1$, $b = 2$, that $(A, +)$ is a group whose identity element is 0. The postulate F 1 is verified. The postulate F 2 is readily checked from the multiplication table. (Be sure that the associative law is verified.)

As far as F 3 is concerned we may put aside the case where $a = 0$ since we know that the product 0 and any element of $A$ is 0. Further since multiplication is

commutative, it suffices to consider only the first of the two distributive laws. The check may be tabulated as follows:

| a | b | c | a · (b + c) | a · b + a · c |
|---|---|---|---|---|
| 1 | 0 | 0 | 1 · 0 = 0 | 0 + 0 = 0 |
| 1 | 0 | 1 | 1 · 1 = 1 | 0 + 1 = 1 |
| 1 | 0 | 2 | 1 · 2 = 2 | 0 + 2 = 2 |
| 1 | 1 | 0 | 1 · 1 = 1 | 1 + 0 = 1 |
| 1 | 1 | 1 | 1 · 2 = 2 | 1 + 1 = 2 |
| 1 | 1 | 2 | 1 · 0 = 0 | 1 + 2 = 0 |
| 1 | 2 | 0 | 1 · 2 = 2 | 2 + 0 = 2 |
| 1 | 2 | 1 | 1 · 0 = 0 | 2 + 1 = 0 |
| 1 | 2 | 2 | 1 · 1 = 1 | 2 + 2 = 1 |
| 2 | 0 | 0 | 2 · 0 = 0 | 0 + 0 = 0 |
| 2 | 0 | 1 | 2 · 1 = 2 | 0 + 2 = 2 |
| 2 | 0 | 2 | 2 · 2 = 1 | 0 + 1 = 1 |
| 2 | 1 | 0 | 2 · 1 = 2 | 2 + 0 = 2 |
| 2 | 1 | 1 | 2 · 2 = 1 | 2 + 2 = 1 |
| 2 | 1 | 2 | 2 · 0 = 0 | 2 + 1 = 0 |
| 2 | 2 | 0 | 2 · 2 = 1 | 1 + 0 = 1 |
| 2 | 2 | 1 | 2 · 0 = 0 | 1 + 2 = 0 |
| 2 | 2 | 2 | 2 · 1 = 2 | 1 + 1 = 2 |

This is, quite frankly, tedious. If the division algorithm has been developed, as well as the result that if a prime number divides a product of integers it divides one of the factors, it is not hard to generalize this exercise to the case where 3 is replaced by an arbitrary prime p, A is replaced by {0,1,..., p - 1} and "addition" and

"multiplication" are defined as in the exercise save that we operate with remainders obtained on division by p. If p is replaced by a natural number which is not a prime, the resulting structure is not a field.

5. The verification of F 1 and F 2 is immediate. Cf. Ex. 15-4, No. 3. The additive identity is a and the multiplicative identity is b. Note that B consists simply of the element b. It suffices to verify

$$b(c_1 + c_2) = bc_1 + bc_2, \qquad c_1, c_2 \in A,$$

to be assured that F 3 holds. Since b = 1,

$$b(c_1 + c_2) = c_1 + c_2$$

and

$$bc_1 + bc_2 = c_1 + c_2.$$

Exercises 15-7; Answers.

1. From our formulas for sum and product we see that the usual addition and multiplication define operations in A. The difference of two elements of A is an element of A, as is easily checked. We have seen that the same holds true for quotients of elements of A. The commutative, associative and distributive laws hold for (A, +, · ) since they hold for the real number system. The verification of the field postulates is now routine.

2. The details parallel those of the first exercise and are readily furnished.

3. Suppose that x is a real number belonging to both A and B. Since $x \in A$, $x = a + b\sqrt{2}$ where a and b are rational. Since $x \in B$, $x = c + d\sqrt{2}$ where c and d are rational. It is essential to recall that $\sqrt{2}$ and $\sqrt{3}$

are both irrational. We start with the equality

$$a + b\sqrt{2} = c + d\sqrt{3}$$

and draw the consequences.

Case 1. $d = 0$. Here $x$ is rational.

Case 2. $d \neq 0$. Here we conclude that

$$\sqrt{3} = \frac{a - c}{d} + \frac{b}{d}\sqrt{2},$$

that is, $\sqrt{3}$ is of the form

$$\alpha + \beta\sqrt{2}$$

where $\alpha$ and $\beta$ are both rational. On taking squares, we have

$$3 = (\sqrt{3})^2 = (\alpha + \beta\sqrt{2})^2$$
$$= \alpha^2 + 2\beta^2 + (2\alpha\beta)\sqrt{2}.$$

Since

$$3 = 3 + 0\sqrt{2},$$

we conclude, by the uniqueness property established in Section 15-7 concerning the representation of the members of $\dot{A}$ in the form $a + b\sqrt{2}$, $a$, $b$ rational, that

$$3 = \alpha^2 + 2\beta^2$$

and

$$0 = 2\alpha\beta .$$

Now $\beta \neq 0$ since $\sqrt{3}$ is irrational. Hence from $0 = 2\alpha\beta$, we conclude $\alpha = 0$ and

$$(**) \qquad 3 = 2\beta^2.$$

At this point we make use of the fact that $\beta$ may be written in the form $p/q$ where $p$ and $q$ are natural numbers which are not both divisible by a natural number greater than one. In particular, $p$ and $q$ cannot both be even. From $(**)$ we obtain

$$3 = 2(\frac{p}{q})^2$$

and hence

$$(***) \qquad 3q^2 = 2p^2.$$

Now $q$ must be even, otherwise the left-hand side of (***) would be odd and the right-hand side even. Hence $q = 2r$, where $r$ is a natural number. From (***) we obtain

$$3(2r)^2 = 2p^2,$$

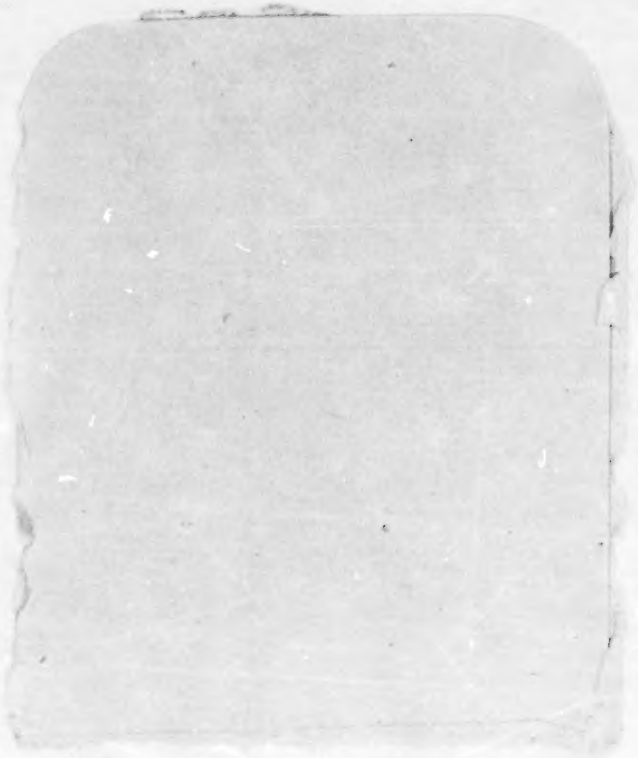
and hence

$$6r^2 = p^2.$$

We now see that $p$ is even. This is impossible, for $p$ is odd. Hence the hypothesis $d \neq 0$ must be rejected.

**Conclusion:** $x$ is rational; i.e., $A \cap B \subset Q$.

Since $Q \subset A \cap B$, we have $Q = A \cap B$.



Below are listed the names of all those who participated in any of the writing sessions at which the following SMSG texts were prepared: *First Course in Algebra, Geometry, Intermediate Mathematics, Elementary Functions,* and *Introduction to Matrix Algebra.*

H. W. Alexander, Earlham College
F. B. Allen, Lyons Township High School, La Grange, Illinois
Alexander Beck, Olney High School, Philadelphia, Pennsylvania
E. F. Beckenbach, University of California at Los Angeles
E. G. Begle, School Mathematics Study Group, Yale University
Paul Berg, Stanford University
Emil Berger, Monroe High School, St. Paul, Minnesota
Arthur Bernhart, University of Oklahoma
R. H. Bing, University of Wisconsin
A. L. Blakers, University of Western Australia
A. A. Blank, New York University
Shirley Boselly, Franklin High School, Seattle, Washington
K. E. Brown, Department of Health, Education, and Welfare, Washington, D.C.
J. M. Calloway, Carleton College
Hope Chipman, University High School, Ann Arbor, Michigan
R. R. Christian, University of British Columbia
R. J. Clark, St. Paul's School, Concord, New Hampshire
P. H. Daus, University of California at Los Angeles
R. B. Davis, Syracuse University
Charles DePrima, California Institute of Technology
Mary Dolciani, Hunter College
Edwin C. Douglas, The Taft School, Watertown, Connecticut
Floyd Downs, East High School, Denver, Colorado
E. A. Dudley, North Haven High School, North Haven, Connecticut
Lincoln Durst, The Rice Institute
Florence Elder, West Hempstead High School, West Hempstead, New York
W. E. Ferguson, Newton High School, Newtonville, Massachusetts
N. J. Fine, University of Pennsylvania
Joyce D. Fontaine, North Haven High School, North Haven, Connecticut
F. L. Friedman, Massachusetts Institute of Technology
Esther Gassett, Claremore High School, Claremore, Oklahoma
R. K. Getoor, University of Washington
V. H. Haag, Franklin and Marshall College
R. R. Hartman, Edina-Morningside Senior High School, Edina, Minnesota
M. H. Heins, University of Illinois
Edwin Hewitt, University of Washington
Martha Hildebrandt, Proviso Township High School, Maywood, Illinois
R. C. Jurgensen, Culver Military Academy, Culver, Indiana
Joseph Lehner, Michigan State University
Marguerite Lehr, Bryn Mawr College
Kenneth Leisenring, University of Michigan
Howard Levi, Columbia University
Eunice Lewis, Laboratory High School, University of Oklahoma
M. A. Linton, William Penn Charter School, Philadelphia, Pennsylvania
A. E. Livingston, University of Washington
L. H. Loomis, Harvard University
R. V. Lynch, Phillips Exeter Academy, Exeter, New Hampshire
W. K. McNabb, Hockaday School, Dallas, Texas
K. G. Michaels, North Haven High School, North Haven, Connecticut
E. E. Moise, University of Michigan
E. P. Northrop, University of Chicago
O. J. Peterson, Kansas State Teachers College, Emporia, Kansas
B. J. Pettis, University of North Carolina
R. S. Pieters, Phillips Academy, Andover, Mass.
H. O. Pollak, Bell Telephone Laboratories
Walter Prenowitz, Brooklyn College
G. B. Price, University of Kansas
A. L. Putnam, University of Chicago
Persis O. Redgrave, Norwich Free Academy, Norwich, Connecticut
Mina Rees, Hunter College
D. E. Richmond, Williams College
C. E. Rickart, Yale University
Harry Ruderman, Hunter College High School, New York City
J. T. Schwartz, New York University
O. E. Stanaitis, St. Olaf College
Robert Starkey, Cubberley High Schools, Palo Alto, California
Phillip Stucky, Roosevelt High School, Seattle, Washington
Henry Swain, New Trier Township High School, Winnetka, Illinois
Henry Syer, Kent School, Kent, Connecticut
G. B. Thomas, Massachusetts Institute of Technology
A. W. Tucker, Princeton University
H. E. Vaughan, University of Illinois
John Wagner, University of Texas
R. J. Walker, Cornell University
A. D. Wallace, Tulane University
E. L. Walters, William Penn Senior High School, York, Pennsylvania
Warren White, North High School, Sheboygan, Wisconsin
D. V. Widder, Harvard University
William Wooton, Pierce Junior College, Woodland Hills, California
J. H. Zant, Oklahoma State University